Analysis of M/{D_n}/1 retrial queue with second optional service, under two disciplines

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***Abstract* - In this paper a single server retrial queue with second optional service is considered in which each customer takes discrete service time on value D_j with probability of p_j for obligatory service and F_k with probability of q_k for second optional service that each customer need this service with probability of θ. We also consider two disciplines for servicing customers; in first discipline, customers receive second optional service (if it is requested) immediately after obligatory service, in second discipline, if second optional service is requested, have to leave service area after obligatory service completion to retrial orbit, and try to be serviced after a exponentially distributed time. The joint steady-state distribution of the state of server and the number of customers in orbit is derived for both disciplines.**

***Keywords* - Retrial queue; Feedback; Probability generating function; Steady State distribution**

I. Introduction

Retrial queue is characterized by the assumption that customers who cannot receive service (due to balking, finite capacity, impatience, etc.) leave the service area and try to obtain service after randomly distributed time. These unsatisfied customers are called retrial customers and between trails, they join a pool called orbit. Cohen [9] discussed retrial queue for the first time on 1957. Retrial Queueing Systems introduced widely in [4, 11]. For a review of main results on retrial queue, see [2, 3, 5, 6].

Most of the papers have considered retrial queues with continues service time, and only some of these papers have discussed discrete service time, see [1, 17, 19]. This paper deals with special case of M/{D_n}/1 retrial queue which is analyzed by Wu and Ke [13]; it is assumed that some customers need to be serviced twice. First service is called obligatory service and the other one is called second optional service which is required with probability of θ. This model can be considered as $M/\{D_n\}/1$ retrial queue with feedback, since some customers return to the system after obligatory receiving service. Kumar [14,15] and Choudhury [8] studied M/G/1 retrial queue with feedback.

For stability condition the reader is referred to [13, 18]. In this model, since there is no waiting line in front of the server, two disciplines are considered for reservicing customers on the base of [16]; in the first policy customers being reserviced immediately after obligatory service completion, however, in the second policy the customers enter to the orbit and make another trail after a time to receive second optional service. One application of this model is Internet packet modelling.

This paper is organized as follows. In section 2, model description with egrodicity condition is given. In section 3, the steady stated distributions of the state of the server and the number of customers in the orbit are derived for each discipline. Subsection 3.1 specialized for first discipline, and subsection 3.2 for second one.

II. Model Description

The authors consider a $M/\{D_n\}/1$ retrial queue with second optional service in which primary customers arrive from outside the system according to a Poisson processes with rate λ. It assumed that there is no waiting room in front of the server, therefore if an arriving customer finds the server busy, enters a retrial orbit and becomes a retrial customer. Otherwise, if the server is found idle, the arriving primary customer is received the obligatory service immediately, and after completion, the customer leaves the system with probably of 1-θ and needs to be serviced again with probability of θ. Retrial customers try to receive the service independently of each other at the time exponentially distributed with rate α and keep making retrial until they obtain service. The obligatory service time, for whether primary or retrial customers, is discrete value of D_j with probability of $p_j, j = 1, 2, \ldots$ and $\sum_{j=1}^{\infty} p_j = 1$. However, second optional service time is discrete value of F_k with probability of $q_k, k = 1, 2, \ldots$ and $\sum_{k=1}^{\infty} q_k = 1$. Interarrival, interretrial and service times are mutually independent. Following describes two disciplines for receiving the second optional service.

In discipline I, after the customer received the obligatory service, the second optional service, if it is required, begins immediately. The server becomes idle unless no other customers, whether primary or retrial, arrive. Therefore, commence rate of primary and retrial customers are λ and α, respectively. However, the service time changes to $D_j + I_\theta F_k$ in which

$$I_\theta = \begin{cases} 1 & \textit{if the customer requiers second optional service} \\ 0 & \textit{otherwise} \end{cases}.$$

In discipline II, after the customer received the obligatory service, goes to retrial orbit if the second optional service is required. Otherwise, the customer leaves the system with probability of 1-θ Therefore, commence rate of primary customers is λ and since $\theta \sum_{j=1}^{\infty} D_j p_j$ of them enter retrial orbit on average, commence rate of retrial customers is $(\theta \sum_{j=1}^{\infty} D_j p_j + i)\alpha$.

Furthermore, according to mentioned explanations, $\sum_{j=1}^{\infty} D_j\, p_j + I_\theta \sum_{k=1}^{\infty} F_k q_k$ customers are serviced on average in both disciplines.

Ergodicity condition

Theorem 1: the necessary and sufficient condition for the system stability is $\lambda(\sum_{j=1}^{\infty} D_j p_j + I_\theta \sum_{k=1}^{\infty} F_k q_k) < 1$.

Proof: Assume that system is stable. Hence, "traffic intensity" should be lower than one. Thus, $\lambda(\sum_{j=1}^{\infty} D_j p_j + I_\theta \sum_{k=1}^{\infty} F_k q_k) < 1$. Conversely, let $\lambda(\sum_{j=1}^{\infty} D_j p_j + I_\theta \sum_{k=1}^{\infty} F_k q_k) < 1$, it is obvious that each customers return to the system with probability of θ. Assume that N_n $(n = 1, 2, \dots)$ is the number of customers in the retrial orbit at the nth departure point. N_n is ergodic due to [19].

III. STEADY STATE DISTRIBUTION OF THE SYSTEM

In this section, we will find the joint steady state distribution of the state of the server and the number of customers in the orbit in departure of nth customer.

At time t, let $C_l(t)$ represent the server state. $C_0(t) = 0$ means that the server is idle, $C_1(t) = j$ means that the server is busy with the service time $D_j, j = 1, 2, \dots$, $C_2(t) = k$ means the server is busy with the service time $F_k, k = 1, 2, \dots$. Let $N(t)$ represent the number of customers in the orbit. Since the service times are discrete, the stochastic process $\{C_l(t), N(t) : l = 0,1,2$ و $t \geq 0\}$ is not necessarily Markovian. So, a supplementary variable $\xi(t)$ is defined as the elapsed service time. Then the stochastic process $\{C_l(t), N(t), \xi(t) : l = 0,1,2$ و $t \geq 0\}$ is a Morkov process. In addition, we define the random variable $C_l^*(t)$ for $l = 1, 2$ in which $C_1^*(t) = j$ means the most recent service time completed up to time t takes on the value $D_j\ for\ j = 1, 2, \dots$, and $C_2^*(t) = k$ means the most recent service time completed up to time t takes on the value $F_k\ for\ k = 1, 2, \dots$.

According to above explanations, following probabilities can be defined.

$(i)\ P_i(t) = P(C_0(t) = 0, N(t) = i), \quad i = 0,1, \dots$

$(ii) P_{(i,j)}(t) = P(C_0(t) = 0, C_1^*(t) = j, N(t) = i), \quad i = 1, 2, \dots; j = 1, 2, \dots$

$(iii) R_{(i,k)}(t) = P(C_0(t) = 0, C_2^*(t) = k, N(t) = i\,), \quad i = 1, 2, \dots; k = 1, 2, \dots$

$(iv) Q_{(i,j)}(t, x) = P\big(C_1(t) = j, N(t) = i, x < \xi(t) < x + dx\big), \quad i = 1, 2, \dots; j = 1, 2, \dots; x \geq 0.$

$(v) S_{(i,k)}(t, x) = P\big(C_2(t) = k, N(t) = i, x < \xi(t) < x + dx\big), \quad i = 1, 2, \dots; k = 1, 2, \dots; x \geq 0.$

In relation to above probabilities, we have

$$P_i(t) = \sum_{j=1}^{\infty} P_{(i,j)}(t) + \sum_{k=1}^{\infty} R_{(i,k)}(t), \quad i = 0,1, \dots$$

Discipline I

Lemma 1: assume that the customer leaves the system after completion of obligatory service, and Δt is a very small time length. Then, for $i = 1, 2, \dots; j = 1, 2, \dots$

$$P_{(i,j)}(t + \Delta t) = P_{(i,j)}(t)\big(1 - (\lambda + i\alpha)\Delta t\big) + \sum_{i'=0}^{i} \left[\lambda \Delta t p_j \frac{(\lambda D_j)^{i-i'}}{(i-i')!} e^{-\lambda D_j} + i'\alpha \Delta t p_j \frac{(\lambda D_j)^{i-i'+1}}{(i-i'+1)!} e^{-\lambda D_j}\right] P_{(i')}(t - D_j) + (i+1)\alpha \Delta t (p_j) e^{-\lambda D_j} P_{(i+1)}(t - D_j) + o(\Delta t)$$

It is worth to mention that all terms in the right hand side of above equations should be multiplied by $(1 - \theta)$, since in this case, the customer leaves the system with the probability of $(1 - \theta)$.

Proof: the event $(C_0(t + \Delta t) = 0, C_1^*(t + \Delta t) = j, N(t + \Delta t) = i)$ occurs if

1) The event $(C_0(t) = 0, C_1^*(t) = j, N(t) = i)$ occurs, and the customer left the system, and no primary or retrial customer arrive between the time t and $t + \Delta t$.
2) The event $\big(C_1(t) = j,\ N(t) = i, D_j - \Delta t < \ \xi(t) < D_j\big)$ occurs and this occurs if
   a) The event $\big(C_0(t - D_j) = 0, N(t - D_j) = i'\big)$ occurs for $i' \in \{0, 1, \dots, i\}$ and a primary customer arrives between the time $t - D_j$ and $t - D_j + \Delta t$ and begins to be serviced on the value of D_j and the customer leaves the system after obligatory service completion, and $i - i'$ primary customers arrive during its service, or
   b) The event $\big(C_0(t - D_j) = 0, N(t - D_j) = i'\big)$ occurs for $i' \in \{0, 1, \dots, i\}$ and a retrial customer arrives between the time $t - D_j$ and $t - D_j + \Delta t$ and begins to be serviced on the value of D_j and the customer leaves the system after obligatory service completion, and $i - i' + 1$ primary customers arrive during its service, or
   c) The event $\big(C_0(t - D_j) = 0, N(t - D_j) = i + 1\big)$ occurs for $i = 1, 2, \dots$ and a retrial customer arrives between the time $t - D_j$ and $t - D_j + \Delta t$ and begins to be serviced on the value of D_j and the customer leaves the system after obligatory service completion, and no primary customer arrives during its service. □

Thus,

$$\frac{d}{dt}P_{(i,j)}(t) = \lim_{\Delta t \to 0} \frac{P_{(i,j)}(t+\Delta t) - P_{(i,j)}(t)}{\Delta t}$$
$$= -(\lambda + i\alpha)P_{(i,j)}(t) + \sum_{i'=0}^{i}\left[\lambda p_j \frac{(\lambda D_j)^{i-i'}}{(i-i')!}e^{-\lambda D_j} + i'\alpha p_j \frac{(\lambda D_j)^{i-i'+1}}{(i-i'+1)!}e^{-\lambda D_j}\right]P_{(i')}(t-D_j)$$
$$+ (i+1)\alpha(p_j)e^{-\lambda D_j}P_{(i+1)}(t-D_j)$$

Lemma 2: Assume that the customer requires the second optional service and remains in the system after obligatory service completion, and Δt is a very small time length. Then, for $i = 1, 2, \dots; k = 1, 2, \dots$

$$R_{(i,k)}(t+\Delta t) = R_{(i,k)}(t)\big(1 - (\lambda + i\alpha)\Delta t\big) + \sum_{i'=0}^{i}\lambda \Delta t q_k \frac{(\lambda F_k)^{i-i'}}{(i-i')!}e^{-\lambda F_k}P_{(i')}(t-F_k) + o(\Delta t)$$

Note that all terms in the right hand side of above equations should be multiplied by θ, since in this case, the customer remains in the system with the probability of θ.

Proof: Note that $R_{(i,k)}(t+\Delta t) = P(C_0(t+\Delta t) = 0, C_2^*(t+\Delta t) = k, N(t+\Delta t) = i)$. The event $(C_0(t+\Delta t) = 0, C_2^*(t)(t+\Delta t) = k, N(t+\Delta t) = i)$ occurs when

1) The event $(C(t) = 0, C^*(t) = k, N(t) = i)$ occurs and no primary and retrial customers arrive between time t and $t+\Delta t$.
2) The event $\big(C(t) = k,\ N(t) = i, F_k - \Delta t < \xi(t) < F_k\big)$ occurs. And this occurs if $(C(t-F_k) = 0, N(t-F_k) = i')$ occurs for $i' \in \{0, 1, \dots, i\}$ and a customer whose obligatory service time was D_j, begins second optional service, after obligatory service completion, between time $t - F_k$ and $t - F_k + \Delta t$ in which service time is F_k, and $i - i'$ primary customers arrive during its service time. □

Thus,

$$\frac{d}{dt}R_{(i,j)}(t) = \lim_{\Delta t \to 0}\frac{R_{(i,j)}(t+\Delta t) - R_{(i,j)}(t)}{\Delta t} = -(\lambda + i\alpha)P_{(i,j)}(t) + \sum_{i'=0}^{i}\lambda q_k \frac{(\lambda F_k)^{i-i'}}{(i-i')!}e^{-\lambda F_k}P_{(i')}(t-F_k)$$

We have also following equations,

$$\frac{\partial}{\partial t}Q_{(i,j)}(t,x) + \frac{\partial}{\partial x}Q_{(i,j)}(t,x) = -\lambda Q_{(i,j)}(t,x) + \lambda Q_{(i-1,j)}(t,x), \qquad i = 0,1,\dots, j = 1,2,\dots$$
$$\frac{\partial}{\partial t}S_{(i,k)}(t,x) + \frac{\partial}{\partial x}S_{(i,k)}(t,x) = -\lambda S_{(i,k)}(t,x) + \lambda S_{(i-1,k)}(t,x), \qquad i = 0,1,\dots, k = 1,2,\dots$$
$$\frac{\partial}{\partial t}Q_{(0,j)}(t,x) + \frac{\partial}{\partial x}Q_{(0,j)}(t,x) = -\lambda Q_{(0,j)}(t,x), \qquad j = 1,2,\dots$$
$$\frac{\partial}{\partial t}S_{(0,k)}(t,x) + \frac{\partial}{\partial x}S_{(0,k)}(t,x) = -\lambda S_{(0,k)}(t,x), \qquad k = 1,2,\dots$$
$$Q_{(i,j)}(t,0) = \lambda p_j\, P_i(t) + (i+1)\alpha p_j\, P_{i+1}(t), \qquad i = 0,1,\dots, j = 1,2,\dots$$
$$S_{(i,k)}(t,0) = \lambda q_k\, P_i(t), \qquad i = 0,1,\dots, k = 1,2,\dots \quad (8)$$
$$\sum_{i=0}^{\infty}P_{(i)}(t) + \sum_{i=0}^{\infty}\left[\sum_{j=1}^{\infty}\int_0^{D_j}Q_{(i,j)}(t,x) + \sum_{k=1}^{\infty}\int_0^{F_k}S_{(i,k)}(t,x)\right] = 1$$

Now, we assume that system is stable. Then,

$$(\lambda + i\alpha)P_{(i,j)} = \sum_{i'=0}^{i}\left[\lambda p_j \frac{(\lambda D_j)^{i-i'}}{(i-i')!}e^{-\lambda D_j} + i'\alpha p_j \frac{(\lambda D_j)^{i-i'+1}}{(i-i'+1)!}e^{-\lambda D_j}\right]P_{(i')}$$
$$+ (i+1)\alpha(p_j)e^{-\lambda D_j}P_{(i+1)}, \quad , \qquad i = 0,1,\dots, j = 1,2,\dots \qquad (1)$$

and

$$(\lambda + i\alpha)R_{(i,k)} = \sum_{i'=0}^{i}\lambda q_k \frac{(\lambda F_k)^{i-i'}}{(i-i')!}e^{-\lambda F_k}P_{(i')}, \quad i = 0,1,\dots, k = 1,2,\dots \qquad (2)$$
$$\frac{\partial}{\partial x}Q_{(i,j)}(x) = -\lambda Q_{(i,j)}(x) + \lambda Q_{(i-1,j)}(x), \qquad i = 0,1,\dots, j = 1,2,\dots \quad (3)$$
$$\frac{\partial}{\partial x}S_{(i,k)}(x) = -\lambda S_{(i,k)}(x) + \lambda S_{(i-1,k)}(x), \qquad i = 0,1,\dots, k = 1,2,\dots \quad (4)$$
$$Q_{(i,j)}(0) = \lambda p_j\, P_i + (i+1)\alpha p_j\, P_{i+1}, \qquad i = 0,1,\dots, j = 1,2,\dots \quad (5)$$
$$S_{(i,k)}(0) = \lambda q_k\, P_i, \qquad i = 0,1,\dots, k = 1,2,\dots \quad (6)$$
$$\sum_{i=0}^{\infty}P_{(i)} + \sum_{i=0}^{\infty}\left[\sum_{j=1}^{\infty}\int_0^{D_j}Q_{(i,j)}(x) + \sum_{k=1}^{\infty}\int_0^{F_k}Q_{(i,k)}(x)\right] = 1$$

In order to solve the above system of equations, we will find the probability generating function.

First of all, we define $P(z)=\sum_{i=0}^{\infty}P_{(i)}\,z^i$ and multiply both sides of equations (1) and (2) by z^i and sum over i,j and k. Then,

$$\left(\lambda+\alpha z\frac{d}{dz}\right)P(z)$$
$$=(1-\theta)\sum_{j=1}^{\infty}\lambda p_j e^{-\lambda D_j(1-z)}P(z)+(1-\theta)\alpha\sum_{j=1}^{\infty}p_j e^{-\lambda D_j(1-z)}\frac{d}{dz}P(z)+(1-\theta)\alpha\sum_{j=1}^{\infty}p_j e^{-\lambda D_j}\frac{d}{dz}P(z)$$
$$+\theta\sum_{j=1}^{\infty}\lambda q_k e^{-\lambda F_k(1-z)}P(z)$$

Following results are obtained by solving above linier differentiation equation.

$$P(z)=Ce^{-\int a(u)du} \quad (7)$$

In which

$$a(u)=\frac{(1-\theta)\sum_{j=1}^{\infty}\lambda p_j e^{-\lambda D_j(1-u)}+\theta\sum_{j=1}^{\infty}\lambda q_k e^{-\lambda F_k(1-u)}-\lambda}{(1-\theta)\alpha\left(\sum_{j=1}^{\infty}p_j e^{-\lambda D_j(1-u)}+\sum_{j=1}^{\infty}p_j e^{-\lambda D_j}\right)-\alpha u}$$

And

$$C=P_0=\left(1-\lambda\left(\sum_{j=1}^{\infty}D_j\,p_j+I_\theta\sum_{k=1}^{\infty}F_k q_k\right)\right)\left(exp\int_0^1 a(u)du\right)$$

When both sides of (3), (4), (5), (6) are multiplied by z^i and summed over i, we obtain

$$\left(\frac{\partial}{\partial x}+\lambda\right)Q_j(z,x)=\lambda z Q_j(z,x),\quad j=1,2,\ldots$$
$$\left(\frac{\partial}{\partial x}+\lambda\right)S_k(z,x)=\lambda z S_k(z,x),\quad k=1,2,\ldots$$
$$Q_j(z,0)=\lambda p_j\,P(z)+\alpha p_j\frac{d}{dz}P(z)$$
$$S_k(z,0)=\lambda q_k P(z)$$
$$Q_j(z,x)=\left(p_j(\lambda-\alpha a(z))\right)P(z)e^{\lambda(z-1)x}\quad (8)$$
$$S_k(z,x)=\lambda q_k P(z)e^{\lambda(z-1)x}\quad (9)$$

(7), (8) and (9) are the steady state distributions of the system for first discipline.

3.2 Discipline II

Lemma 3: For $i=0,1,\ldots$, $j=1,2,\ldots$ and Δt we have,

$$P_{(i,j)}(t+\Delta t)=\left(1-\left(\lambda+\left(\theta\sum_{j=1}^{\infty}D_j p_j+i\right)\alpha\right)\Delta t\right)P_{(i,j)}(t)$$
$$+\sum_{i'=0}^{i}\left[\lambda\Delta t p_j\frac{(\lambda D_j)^{i-i'}}{(i-i')!}e^{-\lambda D_j}+\left(i'-\theta\sum_{j=1}^{\infty}D_j p_j\right)\alpha\Delta t p_j\frac{(\lambda D_j)^{i-i'+1}}{(i-i'+1)!}e^{-\lambda D_j}\right]P_{(i')}(t-D_j)$$
$$+\left(i+1-\theta\sum_{j=1}^{\infty}D_j p_j\right)\alpha\Delta t p_j e^{-\lambda D_j}P_{(i+1)}(t-D_j)+o(\Delta t)$$

Proof: Similar to the proof of Lemma 1. □

Then,

$$\frac{d}{dt}P_{(i,j)}(t)=-\left(\lambda+\left(\theta\sum_{j=1}^{\infty}D_j p_j+i\right)\alpha\right)P_{(i,j)}(t)$$
$$+\sum_{i'=0}^{i}\left[\lambda p_j\frac{(\lambda D_j)^{i-i'}}{(i-i')!}e^{-\lambda D_j}+\left(i'-\theta\sum_{j=1}^{\infty}D_j p_j\right)\alpha p_j\frac{(\lambda D_j)^{i-i'+1}}{(i-i'+1)!}e^{-\lambda D_j}\right]P_{(i')}(t-D_j)$$
$$+\left(i+1-\theta\sum_{j=1}^{\infty}D_j p_j\right)\alpha p_j e^{-\lambda D_j}P_{(i+1)}(t-D_j)$$

Lemma 2: For $i=0,1,\ldots$, $k=1,2,\ldots$ and Δt we have,

$$R_{(i,k)}(t+\Delta t) = R_{(i,k)}(t)\left(1-\left(\lambda+\left(\theta\sum_{j=1}^{\infty}D_jp_j+i\right)\alpha\right)\Delta t\right) + \sum_{i'=0}^{i}\left(\theta\sum_{j=1}^{\infty}D_jp_j\right)\alpha\Delta t q_k\frac{(\lambda F_k)^{i-i'+1}}{(i-i'+1)!}e^{-\lambda F_k}P_{(i')}(t-F_k) + \left(\theta\sum_{j=1}^{\infty}D_jp_j\right)\alpha\Delta t q_k e^{-\lambda F_k}P_{(i+1)}(t-F_k)+o(\Delta t)\ (2)$$

Proof: the event $(C_0(t+\Delta t)=0, C_2^*(t+\Delta t)=k, N(t+\Delta t)=i)$ occurs if,

1) The event $(C_0(t)=0, C_2^*(t)=k, N(t)=i)$ occurs and no primary or retrial customer arrives between the time t and $t+\Delta t$.
2) The event $(C_1(t)=k,\ N(t)=i, F_k-\Delta t<\ \xi(t)<F_k)$ occurs and this occurs if
   a) The event $(C(t-F_k)=0, N(t-F_k)=i')$ occurs for $i'\in\{0,1,\dots,i\}$ and a retrial customer arrives between the time $t-F_k$ and $t-F_k+\Delta t$ and begins to be serviced on the value of F_k, and $i-i'+1$ primary customer arrive during its service, or
   b) The event $(C_0(t-F_k)=0, N(t-F_k)=i+1)$ occurs and a retrial customer arrives between the time $t-F_k$ and $t-F_k+\Delta t$ and begins to be serviced on the value of F_k, and no primary customer arrives during its service. □

Then,

$$\frac{d}{dt}R_{(i,k)}(t) = -\left(\lambda+\left(\theta\sum_{j=1}^{\infty}D_jp_j+i\right)\alpha\right)R_{(i,k)}(t) + \sum_{i'=0}^{i}\left(\theta\sum_{j=1}^{\infty}D_jp_j\right)\alpha q_k\frac{(\lambda F_k)^{i-i'+1}}{(i-i'+1)!}e^{-\lambda F_k}P_{(i')}(t-F_k) + \left(\theta\sum_{j=1}^{\infty}D_jp_j\right)\alpha q_k e^{-\lambda F_k}P_{(i+1)}(t-F_k)$$

If we assume the system is stable, then $\frac{d}{dt}P_{(i,j)}(t)=\frac{d}{dt}R_{(i,k)}(t)=0$, and also

$$\frac{\partial}{\partial t}Q_{(i,j)}(t,x)+\frac{\partial}{\partial x}Q_{(i,j)}(t,x) = -\lambda Q_{(i,j)}(t,x)+\lambda Q_{(i-1,j)}(t,x),\quad i=0,1,\dots,j=1,2,\dots$$

$$\frac{\partial}{\partial t}S_{(i,k)}(t,x)+\frac{\partial}{\partial x}S_{(i,k)}(t,x) = -\lambda S_{(i,k)}(t,x)+\lambda S_{(i-1,k)}(t,x),\quad i=0,1,\dots,k=1,2,\dots$$

$$\frac{\partial}{\partial t}Q_{(0,j)}(t,x)+\frac{\partial}{\partial x}Q_{(0,j)}(t,x) = -\lambda Q_{(0,j)}(t,x),\quad j=1,2,\dots$$

$$\frac{\partial}{\partial t}S_{(0,k)}(t,x)+\frac{\partial}{\partial x}S_{(0,k)}(t,x) = -\lambda S_{(0,k)}(t,x),\quad k=1,2,\dots$$

$$Q_{(i,j)}(t,0) = \lambda p_j\,P_i(t)+(1-\theta)(i+1)\alpha p_j\,P_{i+1}(t)+\theta i\alpha p_j\,P_i(t),\quad i=0,1,\dots,j=1,2,\dots$$

$$S_{(i,k)}(t,0) = i\alpha q_k\,P_i(t)+(i+1)\alpha q_k P_{i+1}(t),\quad i=0,1,\dots,k=1,2,\dots$$

$$\sum_{i=0}^{\infty}P_{(i)}(t)+\sum_{i=0}^{\infty}[\sum_{j=1}^{\infty}\int_0^{D_j}Q_{(i,j)}(t,x)+\sum_{k=1}^{\infty}\int_0^{F_k}S_{(i,k)}(t,x)]=1$$

Now, we assume that system is stable. Then,

$$\left(\lambda+\left(\theta\sum_{j=1}^{\infty}D_jp_j+i\right)\alpha\right)P_{(i,j)} = \sum_{i'=0}^{i}\left[\lambda p_j\frac{(\lambda D_j)^{i-i'}}{(i-i')!}e^{-\lambda D_j}+\left(i'-\theta\sum_{j=1}^{\infty}D_jp_j\right)\alpha p_j\frac{(\lambda D_j)^{i-i'+1}}{(i-i'+1)!}e^{-\lambda D_j}\right]P_{(i')} + \left(i+1-\theta\sum_{j=1}^{\infty}D_jp_j\right)\alpha p_j e^{-\lambda D_j}P_{(i+1)}\quad i=0,1,\dots,j=1,2,\dots\ (10)$$

And

$$\left(\lambda+\left(\theta\sum_{j=1}^{\infty}D_jp_j+i\right)\alpha\right)R_{(i,k)}$$
$$=\sum_{i'=0}^{i}\left(\theta\sum_{j=1}^{\infty}D_jp_j\right)\alpha q_k\frac{(\lambda F_k)^{i-i'+1}}{(i-i'+1)!}e^{-\lambda F_k}P_{(i')}+\left(\theta\sum_{j=1}^{\infty}D_jp_j\right)\alpha q_k e^{-\lambda F_k}P_{(i+1)},\quad i=0,1,\dots,k=1,2,\dots\quad(11)$$

$$\frac{\partial}{\partial x}Q_{(i,j)}(x)=-\lambda Q_{(i,j)}(x)+\lambda Q_{(i-1,j)}(x),\quad i=0,1,\dots,j=1,2,\dots\quad(12)$$
$$\frac{\partial}{\partial x}S_{(i,k)}(x)=-\lambda S_{(i,k)}(x)+\lambda S_{(i-1,k)}(x),\quad i=0,1,\dots,k=1,2,\dots\quad(13)$$
$$Q_{(i,j)}(0)=\lambda p_j\,P_i+(1-\theta)(i+1)\alpha p_j\,P_{i+1}+\theta i\alpha p_j\,P_i,\quad i=0,1,\dots,j=1,2,\dots\quad(14)$$
$$S_{(i,k)}(0)=i\alpha q_k\,P_i+(i+1)\alpha q_kP_{i+1},\quad i=0,1,\dots,k=1,2,\dots\quad(15)$$
$$\sum_{i=0}^{\infty}P_{(i)}+\sum_{i=0}^{\infty}\left[\sum_{j=1}^{\infty}\int_0^{D_j}Q_{(i,j)}(x)+\sum_{k=1}^{\infty}\int_0^{F_k}Q_{(i,k)}(x)\right]=1$$

In order to solve the above system of equations, we will find the probability generating function.
First of all, we define $P(z)=\sum_{i=0}^{\infty}P_{(i)}\,z^i$ and multiply both sides of equations (10) and (11) by z^i and sum over i,j and k. Then

$$\left(\lambda+\left(\theta\sum_{j=1}^{\infty}D_jp_j\right)+\alpha z\right)P(z)$$
$$=\sum_{j=1}^{\infty}\lambda p_je^{-\lambda D_j(1-z)}P(z)+\alpha\sum_{j=1}^{\infty}p_je^{-\lambda D_j(1-z)}\frac{d}{dz}P(z)+\alpha\sum_{j=1}^{\infty}p_je^{-\lambda D_j}\frac{d}{dz}P(z)$$
$$-{}^{\alpha}/_{z}\sum_{j=1}^{\infty}\left(\theta\sum_{j=1}^{\infty}D_jp_j\right)p_je^{-\lambda D_j(1-z)}P(z)-{}^{\alpha}/_{z}\sum_{j=1}^{\infty}\left(\theta\sum_{j=1}^{\infty}D_jp_j\right)p_je^{-\lambda D_j}P(z)$$
$$+{}^{\alpha}/_{z}\left(\theta\sum_{j=1}^{\infty}D_jp_j\right)\left(\sum_{k=1}^{\infty}q_ke^{-\lambda F_k(1-z)}+\sum_{k=1}^{\infty}q_ke^{-\lambda F_k}\right)P(z)$$

Then,

$$P(z)=exp\left(-\int_0^z\frac{A(u)}{B(u)}du\right)P_0\quad(16)$$

In which,

$$A(u)=\left(\lambda+\left(\theta\sum_{j=1}^{\infty}D_jp_j\right)+\alpha u\right)-\sum_{j=1}^{\infty}\lambda p_je^{-\lambda D_j(1-u)}+{}^{\alpha}/_{u}\left(\sum_{j=1}^{\infty}\left(\theta\sum_{j=1}^{\infty}D_jp_j\right)p_je^{-\lambda D_j(1-u)}+\sum_{j=1}^{\infty}\left(\theta\sum_{j=1}^{\infty}D_jp_j\right)p_je^{-\lambda D_j}\right)$$
$$-{}^{\alpha}/_{u}\left(\theta\sum_{j=1}^{\infty}D_jp_j\right)\left(\sum_{k=1}^{\infty}q_ke^{-\lambda F_k(1-u)}+\sum_{k=1}^{\infty}q_ke^{-\lambda F_k}\right)$$
$$B(u)=\alpha\left(\sum_{j=1}^{\infty}p_je^{-\lambda D_j(1-u)}+\alpha\sum_{j=1}^{\infty}p_je^{-\lambda D_j}\right)$$

And,

$$P_0=\left(1-\lambda\left(\sum_{j=1}^{\infty}D_j\,p_j+I_\theta\sum_{k=1}^{\infty}F_kq_k\right)\right)\left(exp\int_0^1\frac{A(u)}{B(u)}du\right)$$

$$Q_j(z,0)=\lambda p_j\,P(z)+(1-\theta)\alpha p_j\frac{d}{dz}\,P(z)+\theta\alpha zp_j\frac{d}{dz}\,P(z)$$
$$S_k(z,0)=\alpha(z+1)q_k\frac{d}{dz}P(z)$$
$$Q_j(z,x)=\left(p_j\left(\lambda+\alpha(\theta(1-z)-1)\frac{A(z)}{B(z)}\right)\right)P(z)e^{\lambda(z-1)x}\quad(17)$$
$$\boldsymbol{S_k(z,x)}=-\alpha(z+1)q_k\frac{A(z)}{B(z)}\boldsymbol{P(z)e^{\lambda(z-1)x}}\quad(18)$$

So, (16), (17) and (18) are the steady state distributions of the system for second discipline.

IV. Conclusion

In this paper, the authors consider $M/\{D_n\}/1$ retrial queue with second optional service. Two different disciplines for reservicing the customers have been discussed: In first discipline, the server gives second optional service immediately after obligatory service, whereas in second discipline, customer enters orbit and becomes retrial customer if the second optional service is required. For each discipline, steady state distributions of the system and probability generating functions are derived.

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