Effect of Laser Heat Source on One Dimensional Generalized Thermoelasticity

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Abstract— In the present paper, an investigation is devoted to study the induced temperature and stress fields in an elastic half space under the purview of generalized thermoelectricity theory of field equations. The half space continuum is considered to be made of an isotropic homogeneous thermoplastic material, the bounding plane surface being subjected to a Non-Gaussian laser pulse. First we employ the Laplace transform to remove the time dependency of the governing equations. Further, we use the finite element method with respect to space coordinate to obtain the numerical solution in Laplace domain. Finally, a numerical method to invert the Laplace transform is used to obtain the numerical solution in space and time domain. The discussion of results are shown by plotting the different graphs.

Index Terms— Thermoelasticity, Laser heating, Finite Element Method, Half Space.

I. INTRODUCTION

Although the classical Fourier heat conduction equation has been applied in the majority of practical engineering applications, there is an important body of problems that requires due consideration of thermomechanical coupling. It is appropriate in these cases to apply the generalized theory of thermoelasticity.

The absence of any elasticity term in the heat conduction equation for uncoupled thermoelasticity appears to be unrealistic, since due to the mechanical loading of an elastic body, the strain so produced causes variation in the temperature field. Moreover, the parabolic type of the heat conduction equation results in an infinite velocity of thermal wave propagation, which also contradicts the actual physical phenomena. Introducing the strain-rate term in the uncoupled heat conduction equation, Biot extended the analysis to incorporate coupled thermoelasticity [1]. In this way, although the first shortcoming was over, there remained the parabolic type partial differential equation of heat conduction, which leads to the paradox of infinite velocity of the thermal wave. To overcome this paradox, generalized thermoelasticity theory was developed subsequently. Due to the advancement of pulsed lasers, fast burst nuclear reactors and particle accelerators, etc. which can supply heat pulses with a very fast time-rise [2,3], generalized thermoelasticity theory is receiving serious attention. The development of the second sound effect has been nicely reviewed by Chandrasekharaiah [4]. At present, mainly two different models of generalized thermoelasticity are being extensively used- one proposed by Lord and Shulman [5] and the other proposed by Green and Lindsay [6]. LS (Lord and Shulman) theory introduces one relaxation time and according to this theory, only Fourier's heat conduction equation is modified. While GL (Green and Lindsay) theory introduces two relaxation times and both the energy equation and the equation of motion are modified.

In the modern physics of laser–matter interactions, different classes of materials likes metals, semiconductors, and dielectrics demonstrate dissimilar character under pulsed laser action [6, 7]. Moreover, the different representatives of the same material class can often behaves differently to the laser irradiation. Among metals there are ductile and brittle ones with the possibility of brittle to ductile transitions [8]. Thus, to achieve high-precision micro-processing, the process strategies have to be developed. Taking into account specific material properties, among which the brittleness and plasticity are of supreme importance [6].

The basic importance of plasticity has been introduced by fs laser induced sub-wavelength structuring of thin gold films with formation of micro-bumps and Nano jets [9, 10]. However, there are only a few papers about concerning the theoretical studies of thermal stresses developed in solids by pulsed laser irradiation (see [11, 12], but post-irradiation stresses can result in lattice deformations and generation of unusual structures such as those obtained in [9, 10].

In the present paper, an investigation is devoted to study the induced temperature and stress fields in an elastic half space under the purview of generalized thermoelasticity theory of field equations. The half space continuum is considered to be made of an isotropic homogeneous thermoelastic material, the bounding plane surface being subjected to a Non-Gaussian laser pulse. First we employ the Laplace transform to remove the time dependency of the governing equations. Further, we use the finite element method with respect to space coordinate to obtain the numerical solution in Laplace domain. Finally, a numerical method to invert the Laplace transform is used to obtain the numerical solution in space and time domain. The discussion of results are shown by plotting the different graphs.

II. Problem Formulation

Equation of motion:

$$\sigma_{ij} = \frac{\partial^2 u_i}{\partial t^2} \tag{1}$$

Stress-strain-temperature relation:

$$\sigma_{ij} = 2\mu e_{ij} + (\lambda e_{kk} - \gamma\theta)\delta_{ij} \tag{2}$$

The equation of heat conduction:

$$(K\theta,_{i}),_{i} = (1+\tau\frac{\partial}{\partial t})\,[\rho c_E \frac{\partial\theta}{\partial t} + T_0\gamma\frac{\partial e}{\partial t} - \rho R] \tag{3}$$

The strain-displacement relation:

$$2e_{ij} = (u_{i,j}+u_{j,i}) \tag{4}$$

From the Eqns. (1) and (2), we get

$$[\mu(u_{i,j}+u_{j,i}) + \lambda u_{kk}\delta_{ij} - \gamma\theta\delta_{ij}]_{,j} = \rho\frac{\partial^2 u_i}{\partial t^2} \tag{5}$$

$$here\ R = \frac{R_a L_0}{\delta t_p^2}\, t e^{-(\frac{h}{2\delta}+\frac{x}{\delta})}\, e^{-\frac{t}{t_p}}$$

We asume that laser heat emits the heat only in the direction of x coordinate. Therefore Eqn. (5) Reduces to the following form:

$$(\boldsymbol{\lambda}+2\mu)\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial(\boldsymbol{\lambda}+\mathbf{2}\boldsymbol{\mu})}{\partial x} - \gamma\frac{\partial\theta}{\partial r} - \theta\frac{\partial\gamma}{\partial r} = \rho\frac{\partial^2 u}{\partial t^2} \tag{6}$$

As for the same reason, the Eqn. (3) reduces to the following form:

$$K\frac{\partial^2\theta}{\partial x^2} + \frac{\partial K}{\partial x}\frac{\partial\theta}{\partial x} = \left(1+\frac{\partial}{\partial t}\right)[\,\rho c_E\frac{\partial\theta}{\partial t} + T_0\gamma\frac{\partial^2 u}{\partial t\partial x} - \rho\frac{R_a L_0}{\delta t_p^2}\, t e^{-\left(\frac{h}{2\delta}+\frac{x}{\delta}\right)}\, e^{-\frac{t}{t_p}}\,] \tag{7}$$

Similarly, Eqn. (2) takes the following form:

$$\sigma_{xx} = (\lambda+2\mu)\frac{\partial u}{\partial x} - \gamma\theta \tag{8}$$

For the computational simplicity of the problem, we introduce the following non-dimensional variables and notation

$$(x',u',h'.\delta') = c_0\eta_0(x,u,h,\delta),\ \theta' = \frac{\gamma}{(\lambda_m+2\mu_m)}\theta,\ (t',t_p',\tau') = c_0^2\eta_0(t,t_p,\tau),$$

$$\sigma_{xx}' = \frac{\sigma_{xx}}{(\lambda_m+2\mu_m)},\ c_0^2 = \frac{(\lambda_m+2\mu_m)}{\rho_m},\ \eta_0 = \frac{\rho_m c_m}{K_m},$$

Using the above notations and variables into the Eqns. (6)-(8) and after droping the primes, we obtain, respectively:

$$\frac{(\lambda+2\mu)}{(\lambda_m+2\mu_m)}\frac{\partial^2\mu}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial(\lambda+2\mu)}{\partial x} - \frac{1}{\gamma_m}\left[\gamma\frac{\partial\theta}{\partial r} + \theta\frac{\partial\gamma}{\partial r}\right] = \frac{\rho c_0^2}{(\lambda_m+2\mu_m)}\frac{\partial^2 u}{\partial t^2} \tag{9}$$

$$\frac{K}{K_m}\frac{\partial^2\theta}{\partial x^2} + \frac{1}{K_m}\frac{\partial K}{\partial x}\frac{\partial\theta}{\partial x} = (1+\tau\frac{\partial}{\partial t})\,[\frac{\rho c_E}{\rho_m c_{E_m}}\frac{\partial\theta}{\partial t} + \frac{\epsilon_1\gamma}{\gamma_m}\frac{\partial^2 u}{\partial t\partial x} - \frac{\epsilon_2\rho}{\rho_m}\, t e^{-\frac{x}{\delta}}\, e^{-\frac{t}{t_p}}] \tag{10}$$

$$\sigma_{xx} = \frac{(\lambda+2\mu)}{(\lambda_m+2\mu_m)}\frac{\partial u}{\partial x} - \frac{\gamma}{\gamma_m}\theta \tag{11}$$

$$\epsilon_1 = \frac{T_0\beta_m^2}{\rho_m c_m(\lambda_m+2\mu_m)},\ \epsilon_2 = \frac{R_0 L_0\gamma_m}{\delta_p^2 K_m c_0}\, e^{-\frac{x}{2\delta}}\ and\ \frac{\rho c_0^2}{(\lambda_m+2\mu_m)} = \frac{\rho}{\rho_m}$$

After applying the Laplace transform to Eqns. (9)-(11), we have the following equations:

$$\frac{(\boldsymbol{\lambda}+\mathbf{2}\boldsymbol{\mu})}{(\boldsymbol{\lambda_m}+\mathbf{2}\boldsymbol{\mu_m})}\frac{\partial^2\bar{u}}{\partial x^2} + \frac{\partial\bar{u}}{\partial x}\frac{\partial(\lambda+2\mu)}{\partial x} - \frac{1}{\gamma_m}\left(\gamma\frac{\partial\bar{\theta}}{\partial x} + \bar{\theta}\frac{\partial\gamma}{\partial x}\right) = \frac{\rho s^2\bar{u}}{\rho_m} \tag{12}$$

$$\left[\frac{K}{K_m}\frac{\partial^2\bar{\theta}}{\partial x^2} + \frac{1}{K_m}\frac{\partial K}{\partial x}\frac{\partial\bar{\theta}}{\partial x}\right] = (1+\tau s)\left[\frac{\rho c E^s}{\rho_m c_m E_m}\bar{\theta} + \frac{\epsilon_1\gamma_s}{\gamma_m}\frac{\partial\bar{u}}{\partial x} - \frac{\epsilon_2\rho}{\rho_m}\frac{t_p{}^2}{(1+t_p s)^2}e^{\frac{-x}{\delta}}\right] \tag{13}$$

$$\overline{\sigma_{xx}} = \frac{(\boldsymbol{\lambda}+\mathbf{2}\boldsymbol{\mu})}{(\boldsymbol{\lambda_m}+\mathbf{2}\boldsymbol{\mu_m})}\frac{\partial\bar{u}}{\partial x} - \frac{\boldsymbol{\gamma}}{\boldsymbol{\gamma_m}}\bar{\theta} \tag{14}$$

Now we apply the Galerkin finite element method to the Eqns. (12)-(13). Therefore, the geometry of the one dimensional medium is divided into M discrete elements of equal length, h along the increasing x direction. Considering the base element (el), l = 1, 2, ...M , the displacement and temperature fields over the base element are approximated as

$$\bar{\boldsymbol{u}}^{e_i} = \sum_{i=1}^{d} \boldsymbol{N}_i^{e_i}\ \bar{\boldsymbol{U}}_i^{e_i} \text{ and } \bar{\boldsymbol{\theta}}^{e_i} = \sum_{i=1}^{d} \boldsymbol{N}_i^{e_i}\ \bar{\boldsymbol{\theta}}_i^{e_i} \tag{15}$$

Where the $N_i^{e_i}$ denotes the shape function that approximates the displacement and temperature fields in the base element and d is the number of nodes in the base element. $\bar{\boldsymbol{U}}_i^{e_i}$ and $\bar{\boldsymbol{\theta}}_i^{e_i}$ $(\boldsymbol{i} = \boldsymbol{1}, \boldsymbol{2}, \dots, \boldsymbol{d})$ are therefore the nodal values of displacement and temperature, respectively. The arrangement of elements are shown in Fig.(ii). Therefore after employing the finite element method to Eqns. $(12) - (13)$, we obtain

$$\int_{V_{e_l}} \left[\frac{(\lambda+2\mu)}{(\lambda_m+2\mu_m)}\frac{\partial^2\bar{u}}{\partial x^2} + \frac{\partial\bar{u}}{\partial x}\frac{\partial(\lambda+2\mu)}{\partial x} - \frac{1}{\gamma_m}\left(\gamma\frac{\partial\bar{\theta}}{\partial x} + \bar{\theta}\frac{\partial\gamma}{\partial x}\right) - \frac{\rho s^2}{\rho_m}\bar{u}\right] N_i^{e_l}\, dV_{e_l} = 0 \tag{16}$$

$$\int_{V_{e_l}} \left[\frac{K}{K_m}\frac{\partial^2\bar{\theta}}{\partial x^2} + \frac{1}{K_M}\frac{\partial K}{\partial x}\frac{\partial\bar{\theta}}{\partial x} - (1+\tau\, s)\left(\frac{\rho c E s}{\rho_m c E_m}\bar{\theta} + \frac{\epsilon_1\gamma s}{\gamma_m}\frac{\partial\bar{u}}{\partial x} - \frac{\epsilon_2\rho}{\rho_m}\frac{t_p^2}{(1+t_p s)^2} e^{-\frac{x}{\delta}}\right)\right] N_i^{e_l}\, dV_{e_l} = 0 \tag{17}$$

Since for one dimensional problem, $dV_{e_l} = dx$ and integration limit varies from $x = x_l\ to\ x_{l+1}$. Hence we have from the above equations

$$\int_{x_l}^{x_{l+1}} \left[\frac{(\lambda+2\mu)}{(\lambda_m+2\mu_m)}\frac{\partial^2\bar{u}}{\partial x^2} + \frac{\partial\bar{u}}{\partial x}\frac{\partial(\lambda+2\mu)}{\partial x} - \frac{1}{\gamma_m}\left(\gamma\frac{\partial\bar{\theta}}{\partial x} + \bar{\theta}\frac{\partial\gamma}{\partial x}\right) - \frac{\rho s^2}{\rho_m}\bar{u}\right] N_i^{e_l}\, dx = 0 \tag{18}$$

$$\int_{x_l}^{x_{l+1}} \left[\frac{K}{K_m}\frac{\partial^2\bar{\theta}}{\partial x^2} + \frac{1}{K_M}\frac{\partial K}{\partial x}\frac{\partial\bar{\theta}}{\partial x} - (1+\tau\, s)\left(\frac{\rho c E s}{\rho_m c E_m}\bar{\theta} + \frac{\epsilon_1\gamma s}{\gamma_m}\frac{\partial\bar{u}}{\partial x} - \frac{\epsilon_2\rho}{\rho_m}\frac{t_p^2}{(1+t_p s)^2} e^{-\frac{x}{\delta}}\right)\right] N_i^{e_l}\, dx = 0 \tag{19}$$

After applying week formulation to the eqns. $(18) - (19)$ to reduce second derivative and using Eqn. 15, we obtain the following form of above Eqns.

$$\int_{x_l}^{x_{l+1}} \left[\frac{(\lambda+2\mu)}{(\lambda_m+2\mu_m)}\frac{\partial N_i^{e_l}}{\partial x}\frac{\partial\bar{u}}{\partial x} + \frac{1}{\gamma_m}\left(\gamma\frac{\partial\bar{\theta}}{\partial x} + \bar{\theta}\frac{\partial\gamma}{\partial x}\right) N_i^{e_l} + \frac{\rho s^2}{\rho_m} N_i^{e_i}\bar{u}\right] dx = \left[\frac{(\lambda+2\mu)}{(\lambda_m+2\mu_m)} N_i^{e_i}\frac{\partial\bar{u}}{\partial x}\right]_{x=x_l}^{x=x_{l+1}} \tag{20}$$

$$\int_{x_l}^{x_{l+1}} \left[\frac{K}{K_m}\frac{\partial N_i^{e_i}}{\partial x}\frac{\partial\bar{\theta}}{\partial x} + s(1+\tau\, s)\left(\frac{\rho c E s}{\rho_m c E_m} N_i^{e_i}\bar{\theta} + \frac{\epsilon_1\gamma}{\gamma_m} N_i^{e_i}\frac{\partial\bar{u}}{\partial x}\right)\right] dx = \left[\frac{K}{K_m} N_i^{e_i}\frac{\partial\bar{\theta}}{\partial x}\right]_{x=x_l}^{x=x_{l+1}} + m(s)\int_{x_l}^{x_{l+i}} \frac{\rho}{\rho_m} N_i^{e_i} e^{-\frac{x}{\delta}} dx \tag{21}$$

Where $m(s) = \frac{(1+\tau s)\epsilon_2 t_p^2}{(1+t_p s)^2}$

Putting the approximation of temperature and displacement for nodes of the base element (e_l) from Eqn. (15)into Eqns. $(20) - (21)$ and we obtain the system of linear algebraic equations in unknown nodal values $\bar{U}^{e_l}$ and $\bar{\theta}^{e_l}$ as

$$\begin{bmatrix}[A^{e_l}] & [B^{e_l}] \\ [C^{e_l}] & [D^{e_l}]\end{bmatrix}\begin{bmatrix}[\bar{U}^{e_l}] \\ [\bar{\theta}^{e_l}]\end{bmatrix} = \begin{bmatrix}[F^{e_l}] \\ [G^{e_l}]\end{bmatrix} \tag{22}$$

Where the sub-matrices $[A^{e_l}], [B^{e_l}], [C^{e_l}], [D^{e_l}]$, of order $(d \times d)$ and $[F^{e_l}], [G^{e_l}]$, of order $(d \times 1)$, defined for the base element (e_l) are obtained as follows:

$$A_{ij}^{e_l} = \int_{x_l}^{x_{l+1}} \left[\frac{(\lambda+2\mu)}{(\lambda_m+2\mu_m)}\frac{\partial N_i^{e_l}}{\partial x}\frac{\partial N_j^{e_l}}{\partial x} + \frac{\rho s^2}{\rho_m} N_i^{e_l} N_j^{e_l}\right] dx \tag{23}$$

$$B_{ij}^{e_l} = \int_{x_l}^{x_{l+1}} \frac{1}{\beta_m}\left[\beta\frac{\partial N_j^{e_l}}{\partial x} + \frac{\partial\beta}{\partial x} N_j^{e_l}\right] dx \tag{24}$$

$$D_{ij}^{e_l} = \int_{x_l}^{x_{l+1}} \left[\frac{K}{K_m}\frac{\partial N_i^{e_l}}{\partial\xi}\frac{\partial N_j^{e_l}}{\partial\xi} + s(1+\tau\, s)\left(\frac{\rho c E s}{\rho_m c E_m} N_i^{e_l} N_j^{e_l}\right)\right] dx \tag{25}$$

$$F_i^{e_l} = \left[\frac{(\lambda+2\mu)}{(\lambda_m+2\mu_m)} N_i^{e_l}\frac{\partial\bar{u}^{e_l}}{\partial\xi}\right]_{x_l}^{x_{l+1}} \tag{26}$$

$$G_i^{e_l} = \left[\frac{K}{K_m} N_i^{e_l}\frac{\partial\theta}{\partial x}\right]_{x_l}^{x_{l+1}} + m(s)\int_{x_l}^{x_{l+1}} \frac{\rho}{\rho_m} N_i^{e_l} e^{-\frac{x}{\delta}} dx \tag{27}$$

Where, $i, j = 1, 2, \dots, d$.

After assembling the element matrices given in the Eqn. (22) for all the elements, we get a system of $dM + d$ number of linear algebraic equations in unknown nodal values of displacement and temperature. While assembling, we interchange the rows and columns of element matrix so that the terms in the right hand sides of the equations (26) and (27) canceled out by each other between any two adjacent elements, except the first node of the first element and last node of the last element. Further in-canceled terms of the right side of Eqns. (26) and (27) are known and related to the inner and outer boundary conditions as given by the Eqn. (22).

After using Eqns. (17) and (22) we get,

$$\bar{\theta}_1^{e_1} = \frac{\theta^*}{s}, -a(\lambda_c + 2\mu_c)\frac{\partial \bar{u}^{e_l}}{\partial \xi}|_1 = \lambda_c\, \bar{U}_1^{e_1} - \frac{a(\lambda_m+2\mu_m)\beta_c}{\beta_m}\, \bar{\theta}_1^{e_1} \quad (28)$$

$$\frac{\partial \bar{\theta}^{eM}}{\partial \xi}|_d = 0, \bar{U}_d^{eM} = 0 \quad (29)$$

Further, using Eqn. (22), we obtain

$$\frac{\partial \bar{\theta}^{e_1}}{\partial \xi}|_1 = 0, \ and \ \frac{\partial \bar{u}^{eM}}{\partial \xi}|_d = 0 \quad (30)$$

Therefore, the sub-matrices of the global force matrix $\begin{bmatrix}[F]\\ [G]\end{bmatrix}$ are given by

$$[F] = \begin{bmatrix} \lambda_c\, \bar{U}_1^{e_1} & -\frac{a(\lambda_m+2\mu_m)\beta_c}{\beta_m}\, \bar{\theta}_1^{e_1} \\ & 0 \\ & \vdots \\ & 0 \end{bmatrix}, \text{ and } [G] = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (31)$$

Since, in obtained system of algebraic equations, $\bar{\theta}_1^{e_1}$ and $\bar{u}_d^M$ are known to us. Therefore, two rows and two columns intersecting the nodal values $\bar{\theta}_1^{e_1}$ and $\bar{u}_d^M$ are deleted from the global stiffness matrix. At the same time in global force matrix $\begin{bmatrix}[F]\\ [G]\end{bmatrix}$, the sub-matrices will turn accordingly.

III. Numerical Results and Discussions:

We discuss in detail, the finite element method for coupled partial differential equations to analyze the effect of laser heating on the physical quantities, displacement, temperature and stresses for the generalized thermoelasticity proposed by LS theory [5] in case of half space which is supposed to be initially at reference temperature $T_0 = 293$ K and subjected to the boundary conditions as mentioned above . We solve the system of linear algebraic equations numerically discussed in the previous section by using Matlab software.

For the numerical calculations, we used the following values of constants.

E =66.2 GPa , α =10.3 × $10^{-6}\ K^{-1}$, c_E=808.3 $JKg^{-1}\ K^{-1}$, ρ =4410 KgK^{-1} , K =18.1 $K^{-1}\ s^{-1}$, ν =0.321 .

The numerical results for displacement (u), temperature (θ) and stress (σ_{xx}) due to internal laser heat source are displayed in the figures. 1,2,3 respectively.

Fig.1 shows that that the displacement, u is in agreement with the boundary condition and u tends to zero as x tends to large value. Displacement u shows many local maxima and minima within the effective region.

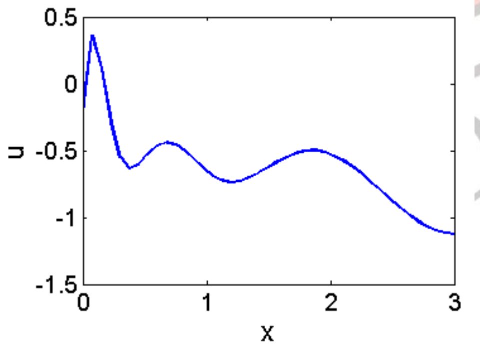

Fig.1: Variation of u vs. x at time t=0.13

Fig.2 shows the variation of temperature, θ vs. space coordinate x. It is evident from the fig.1 that θ starts decreasing with x and finally tends to zero as x tends to large value.

The stress σ_{xx} is shown in the fig.3. It shows that initially stress is compressive in nature and after travelling some distance it comes to be tensile. It also is in agreement with the given boundary condition. Stress, σ_{xx} also tends to zero as x tends to high value.

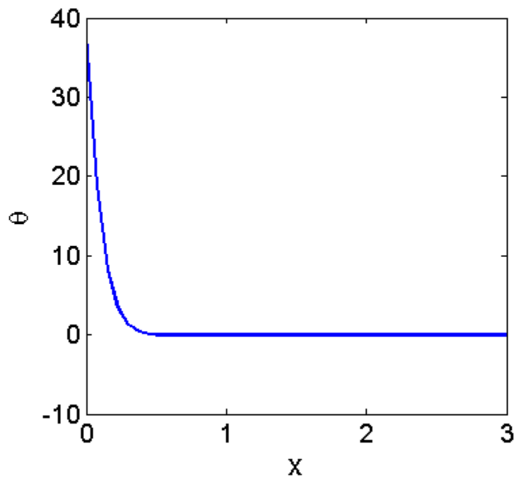

Fig.2: Variation of θ vs. x at time t=0.13

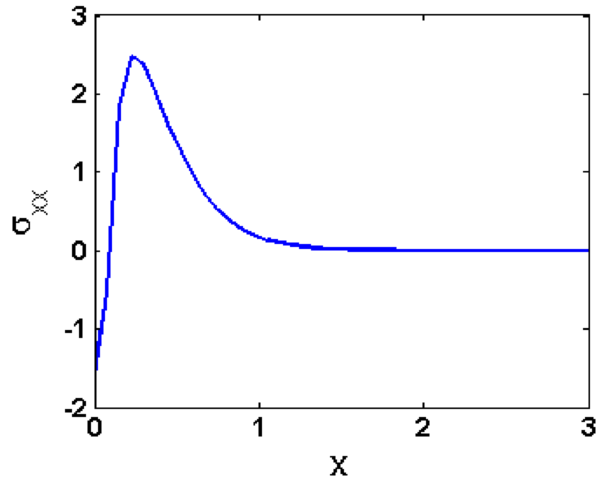

Fig.3: Variation of σ_{xx} vs. x at time t=0.13

IV. References

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