Analysis of Max flow min cut theorem and its generalization

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Abstract - The value of flow is defined by, where s is the source of N. It represents the amount of flow passing from the source to the sink. The maximum flow problem is to maximize | f |, that is, to route as much flow as possible from s to t. An s-t cut C = (S,T) is a partition of V such that s∈S and t∈T. The cut-set of C is the set {(u,v)∈E | u∈S, v∈T}. Note that if the edges in the cut-set of C are removed, | f | = 0. The capacity of an s-t cut is defined by The minimum s-t cut problem is minimizing, that is, to determine S and T such that the capacity of the S-T cut is minimal. In other words, the amount of flow passing through a vertex cannot exceed its capacity. Define an s-t cut to be the set of vertices and edges such that for any path from s to t, the path contains a member of the cut. In this case, the capacity of the cut is the sum the capacity of each edge and vertex in it. In this new definition, the generalized max-flow min-cut theorem states that the maximum value of an s-t flow is equal to the minimum capacity of an s-t cut in the new sense.

Key words: minimizing, partition, vertex, theorem, max-flow.

Introduction

In the most common sense of the term, a graph is an ordered pair G = (V, E) comprising a set V of vertices or nodes together with a set E of edges or lines, which are 2-element subsets of V (i.e., an edge is related with two vertices, and the relation is represented as unordered pair of the vertices with respect to the particular edge). To avoid ambiguity, this type of graph may be described precisely as undirected and simple.

Other senses of graph stem from different conceptions of the edge set. In one more generalized notion, Eis a set together with a relation of incidence that associates with each edge two vertices. In another generalized notion, E is a multi set of unordered pairs of (not necessarily distinct) vertices. Many authors call this type of object a multigraph or pseudograph.

All of these variants and others are described more fully below.

The vertices belonging to an edge are called the ends, endpoints, or end vertices of the edge. A vertex may exist in a graph and not belong to an edge.

V and E are usually taken to be finite, and many of the well-known results are not true (or are rather different) for infinite graphs because many of the arguments fail in the infinite case. The order of a graph is $|V|$ (the number of vertices). A graph's size is the number of edges. The degree of a vertex is the number of edges that connect to it, where an edge that connects to the vertex at both ends (a loop) is counted twice.

For an edge {u, v}, graph theorists usually use the somewhat shorter notation uv.

Adjacency relation

The edges E of an undirected graph G induce a symmetric binary relation ~ on V that is called the adjacency relation of G. Specifically, for each edge {u, v} the vertices u and v are said to be adjacent to one another, which is denoted u ~ v.

Review of Literature

More than one century after Euler's paper on the bridges of Konigsberg and while Listing introduced topology, Cayley was led by the study of particular analytical forms arising from differential calculus to study a particular class of graphs, the trees. This study had many implications in theoretical chemistry. The involved techniques mainly concerned the enumeration of graphs having particular properties. Enumerative graph theory then rose from the results of Cayley and the fundamental results published by Pólya between 1935 and 1937 and the generalization of these by De Bruijn in 1959. Cayley linked his results on trees with the contemporary studies of chemical composition. The fusion of the ideas coming from mathematics with those coming from chemistry is at the origin of a part of the standard terminology of graph theory.

In particular, the term "graph" was introduced by Sylvester in a paper published in 1878 in Nature, where he draws an analogy between "quantic invariants" and "co-variants" of algebra and molecular diagrams:

"Every invariant and co-variant thus becomes expressible by a graph precisely identical with a Kekuléan diagram or chemicograph.I gives a rule for the geometrical multiplication of graphs, i.e. for constructing a graph to the product of in- or co-variants whose separate graphs are given." (Italics as in the original).

In mathematics, graphs are useful in geometry and certain parts of topology, e.g. Knot Theory. Algebraic graph theory has close links with group theory. A graph structure can be extended by assigning a weight to each edge of the graph. Graphs with weights, or weighted graphs, are used to represent structures in which pair wise connections have some numerical values. For example if a graph represents a road network, the weights could represent the length of each road.

The first textbook on graph theory was written by Dénes Kőnig, and published in 1936. A later textbook by Frank Harary, published in 1969, was enormously popular, and enabled mathematicians, chemists, electrical engineers and social scientists to talk to each other. Harary donated all of the royalties to fund the Pólya Prize.
One of the most famous and productive problems of graph theory are the four color problem: "Is it true that any map drawn in the plane may have its regions colored with four colors, in such a way that any two regions having a common border have different colors?" This problem was first posed by Francis Guthrie in 1852 and its first written record is in a letter of De Morgan addressed to Hamilton the same year. Many incorrect proofs have been proposed, including those by Cayley, Kempe, and others. The study and the generalization of this problem by Tait, Heawood, Ramsey and Hadwiger led to the study of the colorings of the graphs embedded on surfaces with arbitrary genus. Tait's reformulation generated a new class of problems, the factorization problems, particularly studied by Petersen and Kőnig. The works of Ramsey on colorations and more specially the results obtained by Turán in 1941 was at the origin of another branch of graph theory, extremal graph theory.
The four color problem remained unsolved for more than a century. In 1969 Heinrich Heesch published a method for solving the problem using computers. A computer-aided proof produced in 1976 by Kenneth Appel and Wolfgang Haken makes fundamental use of the notion of "discharging" developed by Heesch. The proof involved checking the properties of 1,936 configurations by computer, and was not fully accepted at the time due to its complexity. A simpler proof considering only 633 configurations was given twenty years later by Robertson, Seymour, Sanders and Thomas.

Material and Method

n optimization theory, the max-flow min-cut theorem states that in a flow network, the maximum amount of flow passing from the source to the sink is equal to the minimum capacity that when removed in a specific way from the network causes the situation that no flow can pass from the source to the sink.
The max-flow min-cut theorem is a special case of the duality theorem for linear programs and can be used to derive Menger's theorem and the König-Egerváry Theorem.
Let $N = (V, E)$ be a network (directed graph) with s and t being the source and the sink of N respectively.

The capacity of an edge is a mapping c: $E \to R^+$, denoted by c_{uv} or c(u,v). It represents the maximum amount of flow that can pass through an edge.

A flow is a mapping f: $E \to R^+$, denoted by f_{uv} or f(u,v), subject to the following two constraints:

1. $f_{uv} \leq c_{uv}$ for each $(u,v) \in E$ (capacity constraint)
2. $\sum_{u:\,(u,v)\in E} f_{uv} = \sum_{u:\,(v,u)\in E} f_{vu}$

for each $v \in V \setminus \{s,t\}$ (Conservation of flows).

The value of flow is defined by, where s is the source of N. It represents the amount of flow passing from the source to the sink.
The maximum flow problem is to maximize | f |, that is, to route as much flow as possible from s to t.
An s-t cut C = (S,T) is a partition of V such that s∈S and t∈T. The cut-set of C is the set {(u,v)∈E | u∈S, v∈T}. Note that if the edges in the cut-set of C are removed, | f | = 0.

The capacity of an s-t cut is defined by $c(S,T) = \sum_{(u,v)\in S\times T} c_{uv}$.

The minimum s-t cut problem is minimizing, that is, to determine S and T such that the capacity of the S-T cut is minimal.
The max-flow min-cut theorem states
The maximum value of an s-t flow is equal to the minimum capacity over all s-t cuts.

Linear program formulation.

The max-flow problem and min-cut problem can be formulated as two primal-dual linear programs.

Max-flow (Primal)	Min-cut (Dual)
maximize $\lvert f\rvert = \nabla_s$	Minimize $\sum_{(i,j)\in E} c_{ij} d_{ij}$
subject to	subject to

$\sum_{j:(j,i}$ $\nabla_s + \sum_{j:(j,s)}$ $-\nabla_s + \sum_{j:(j,t}$	$d_{ij} - p_i + p_j \geq 0$ $p_s - p_t \geq 1$ $p_i \geq 0$ $d_{ij} \geq 0$

The equality in the max-flow min-cut theorem follows from the strong duality theorem in linear programming, which states that if the primal program has an optimal solution, x*, then the dual program also has an optimal solution, y*, such that the optimal values formed by the two solutions are equal.

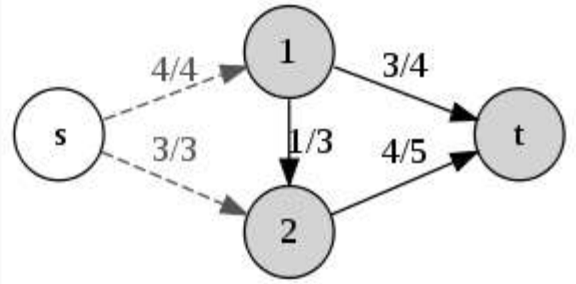

Fig. A network with the value of flow equal to the capacity of an s-t cut

The figure on the right is a network having a value of flow of 7. The vertex in white and the vertices in grey form the subsets S and T of an s-t cut, whose cut-set contains the dashed edges. Since the capacity of the s-t cut is 7, which equals to the value of flow, the max-flow min-cut theorem tells us that the value of flow and the capacity of the s-t cut are both optimal in this network.

Generalized max-flow min-cut theorem

In addition to edge capacity, consider there is capacity at each vertex, that is, a mapping c: $V \rightarrow R^+$, denoted by c(v), such that the flow f has to satisfy not only the capacity constraint and the conservation of flows, but also the vertex capacity constraint

$$\sum_{i \in V} f_{iv} \leq c(v) \quad \text{For each } v \in V \setminus \{s, t\}.$$

Conclusion

In other words, the amount of flow passing through a vertex cannot exceed its capacity. Define an s-t cut to be the set of vertices and edges such that for any path from s to t, the path contains a member of the cut. In this case, the capacity of the cut is the sum the capacity of each edge and vertex in it.

In this new definition, the generalized max-flow min-cut theorem states that the maximum value of an s-t flow is equal to the minimum capacity of an s-t cut in the new sense.

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