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Analytic Solution of Fractional Differential Equation using Fractional Laplace Transform of Hyperbolic Functions

[1]S. Ruban Raj, [2]S. Sahaya Jernith
[1]Associate Professor, [2]Research Scholar
[1] Department of Mathematics,
[1] St. Joseph's College, Trichirapallai, India

***Abstract*— In this paper we introduce some result relate to Fractional Laplace transform with hyperbolic function in order to solve certain fractional differential equation. Mittag –Leffler function plays an important role in the fractional Laplace transform to find the solution of fractional order differential equations. Here we solve some problems related to fractional order differential equation using fractional Laplace Transform with initial conditions.**

***IndexTerm*----- Riemann--Liouville derivative, Mittag--Leffler function, Jumarie Derivative, Fractional Laplace Transforms, Fractional differential equations.**

I. INTRODUCTION

Fractional Calculus arises from a question posed by L'Hospital and Lebnitz in 1965. It is the generalisation of integer-order calculus. Reviewing the history we find that the Fractional Calculus was more interesting topic to mathematicians for a long time in spite of the lack of application back ground. Upcoming years more and more researchers have paid their attention towards Fractional Calculus which are used in real world problems such as Viscoelastic system, dielectric polarization, electromagnetic waves, etc., [8, 9, 5, 7] With the great efforts of researchers there have been rapid developments on the theory of fractional calculus and its applications.

The purpose of this paper is to define the Fractional Laplace Transform via Mittag-Leffler function. Mittag-Leffler function is defined and with its help the Fractional Laplace Transform is also defined. Here we derive some properties of Fractional Laplace Transform based on our definition which is more helpful to find the solution of fractional differential equation with the initial conditions.

II. BASIC DEFINITIONS

Definition 2.1 (Riemann--Liouville definition of fractional derivative). *Let the function $g(t)$ be one time integrable. Then the integro--differential defines Riemann--Liouville fractional derivative* [2]

$${}_bD_t^{\alpha}[g(t)] = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_b^t (t-\tau)^{(n-\alpha)-1} g(\tau)d\tau, \ \alpha > 0$$

This expression is known as the Riemann--Liouville definition of fractional derivative. By this definition, fractional derivative of a constant is non-zero.

Definition 2.2 (Modified Riemann--Liouville (RL)). *To overcome the shortcoming that the fractional derivative of a constant is non--zero, the modification in the definition of the fractional derivative, proposed by Jumarie [5] is described as below*:

$${}_bD_t^{\alpha}[g(t)] = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_b^t (t-\tau)^{-\alpha-1} f(\tau)d\tau, \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_b^t (t-\tau)^{-\alpha}[f(\tau)-f(b)]d\tau, \ \ 0 < \alpha < 1 \\ (f^{(\alpha-m)}(x))^m, m \le \alpha < m+1 \end{cases}$$

Definition 2.3 (Mittag-Leffler Function). *The one--parameter of Mittag--Leffler function* [3], *denoted by $E_\alpha(z)$, is defined by*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \qquad z \in \mathbb{C}, Re(\alpha) > 0 \tag{1}$$

2.1 MITTAG-LEFFLER FUNCTION FOR FRACTIONAL DERIVATIVE

Mittag Leffler function is defined in the form of an infinite series with one parameter [3]

$$E_\beta(\alpha x^\beta) = 1 + \frac{\alpha x^\beta}{\Gamma(1+\beta)} + \frac{\alpha^2 x^{2\beta}}{\Gamma(1+2\beta)} + \frac{\alpha^3 x^{3\beta}}{\Gamma(1+3\beta)} + \cdots \infty$$

Definition 2.4 (Fractional Derivative of Mittag-Leffler Function). *The Jumarie derivative* [3] *of the Mittag-Leffler function* $E_\beta(\alpha x^\beta)$ *is defined as follows*:
Applying term by term Modified RL derivative we get,

$$\begin{aligned} D^\beta[E_\beta(\alpha x^\beta)] &= D^\beta\left[1+\frac{\alpha x^\beta}{\Gamma(1+\beta)}+\frac{\alpha^2x^{2\beta}}{\Gamma(1+2\beta)}+\frac{\alpha^3x^{3\beta}}{\Gamma(1+3\beta)}+\cdots\infty\right] \\ &= \alpha\left[1+\frac{\alpha x^\beta}{\Gamma(1+\beta)}+\frac{\alpha^2x^{2\beta}}{\Gamma(1+2\beta)}+\frac{\alpha^3x^{3\beta}}{\Gamma(1+3\beta)}+\cdots\infty\right] \\ &= \alpha\, E_\beta(\alpha x^\beta) \end{aligned}$$

where β *is the order of the Jumarie derivative of Mittag-Leffler function.*

Definition 2.5 (Fractional Laplace Transform). *If a function* $f(t)$ *is defined for all positive values of the variable* t *and if* $\int_0^\infty E_\beta\,(-s^\beta t^\beta)f(t)(dt)^\beta$ *exists and is equal to* $F(s)$ *, then* $F(s)$ *is called the Fractional Laplace Transform* [4] *of* $f(t)$ *, denoted by the symbol* $L_\beta[f(t)]$. *Hence*

$$L_\beta[f(t)] = \int_0^\infty E_\beta\,(-s^\beta t^\beta)f(t)(dt)^\beta = F(s) \tag{2}$$

The operator L_β *that* $transforms$ $f(t)$ *into* $F(s)$ *is called the Fractional Laplace transform operator.*

Corollary 2.6.

$$L_\beta[E_\beta(at^\beta)] = \frac{1}{s^\beta-a}, s^\beta \neq a$$

Proof.

$$\begin{aligned} L_\beta[E_\beta(at^\beta)] &= L_\beta\left[\sum_{n=0}^{\infty}\frac{a^n t^{\beta n}}{\Gamma(\beta n+1)}\right] = \sum_{n=0}^{\infty}\frac{a^n}{\Gamma(\beta n+1)}L_\beta[t^{\beta n}] \\ &= \sum_{n=0}^{\infty}\frac{a^n}{\Gamma(\beta n+1)}\frac{\Gamma(1+\beta n)}{s^{n\beta+\beta}} \\ &= \sum_{n=0}^{\infty}\frac{a^n}{\Gamma(\beta n+1)}\frac{\Gamma(1+\beta n)}{s^{n\beta+\beta}} \\ &= \frac{1}{s^\beta}\sum_{n=0}^{\infty}\left(\frac{a}{s^\beta}\right)^n \\ &= \frac{1}{s^\beta-a} \end{aligned}$$

□

Corollary 2.7. *Prove that* $L_\beta[f(t)+g(t)] = L_\beta[f(t)] + L[g(t)]$

Proof.

$$\begin{aligned} L_\beta[f(t)+g(t)] &= \int_0^\infty E_\beta\,(-s^\beta t^\beta)[f(t)+g(t)](dt)^\beta \\ &= \int_0^\infty E_\beta\,(-s^\beta t^\beta)f(t)(dt)^\beta + \int_0^\infty E_\beta\,(-s^\beta t^\beta)g(t)(dt)^\beta \\ &= L_\beta[f(t)] + L[g(t)] \end{aligned}$$

□

III. Main Results

Here we define the hyperbolic function of cosine and sine function with the help of Mittag--Leffler function as follows

(a) $cosh_\beta(at^\beta) = \dfrac{E_\beta(at^\beta)+E_\beta(-at^\beta)}{2}$

(b) $sinh_\beta(at^\beta) = \dfrac{E_\beta(at^\beta)-E_\beta(-at^\beta)}{2}$

Result 3.1.

$$L_\beta(cosh_\beta(at^\beta)) = \frac{s^\beta}{s^{2\beta}-a^2}$$

Proof

$$L_\beta(\cosh_\beta(at^\beta)) = L_\beta\left(\frac{E_\beta(at^\beta)+E_\beta(-at^\beta)}{2}\right)$$

$$L_\beta(\cosh_\beta(at^\beta)) = \frac{1}{2}[L_\beta(E_\beta(at^\beta)+L_\beta(E_\beta(-at^\beta)]$$

$$= \frac{1}{2}\left[\frac{1}{s^\beta-a}+\frac{1}{s^\beta+a}\right]$$

$$= \frac{s^\beta}{s^{2\beta}-a^2}$$

□

. **Result 3.2.**

$$L_\beta(\sinh_\beta(at^\beta)) = \frac{a}{s^{2\beta}-a^2}$$

Proof.

$$L_\beta(\sinh_\beta(at^\beta)) = L_\beta\left(\frac{E_\beta(at^\beta)-E_\beta(-at^\beta)}{2}\right)$$

$$L_\beta(\cosh_\beta(at^\beta)) = \frac{1}{2}[L_\beta(E_\beta(at^\beta)-L_\beta(E_\beta(-at^\beta)]$$

$$= \frac{1}{2}\left[\frac{1}{s^\beta-a}-\frac{1}{s^\beta+a}\right]$$

$$= \frac{a}{s^{2\beta}-a^2}$$

□

Example 3.3. *We solve the following homogeneous FDE using Fractional Laplace Transform,*

$$\left(D^{2\left(\frac{1}{4}\right)}-5D^{\frac{1}{4}}+6\right)y(t) = \sinh_{\frac{1}{4}}\left(t^{\frac{1}{4}}\right)$$

where $y(O)=1, y^{\frac{1}{4}}(0)=-1$

Solution: The equation can be written in the form

$$y^{2\left(\frac{1}{4}\right)}-5y^{\left(\frac{1}{4}\right)}+6y = \sinh_{\frac{1}{4}}\left(t^{\frac{1}{4}}\right)$$

Applying fractional Laplace transforms to both sides, we have

$$L_{\frac{1}{4}}\left[y^{2\left(\frac{1}{4}\right)}(0)-5y^{\frac{1}{4}}(0)+6y(0)\right] = L_{\frac{1}{4}}\left[\sinh_{\frac{1}{4}}\left(t^{\frac{1}{4}}\right)\right]$$

$$L_{\frac{1}{4}}\left[y^{2\left(\frac{1}{4}\right)}(0)\right]-5L_{\frac{1}{4}}\left[y^{\frac{1}{4}}(0)\right]+6L_{\frac{1}{4}}[y(0)] = L_{\frac{1}{4}}\left[\sinh_{\frac{1}{4}}\left(t^{\frac{1}{4}}\right)\right]$$

$$s^{\frac{2}{4}}L_{\frac{1}{4}}[y(0)]-s^{\frac{1}{4}}y(0)-y^{\frac{1}{4}}(0)]-5\left[s^{\frac{1}{4}}L_{\frac{1}{4}}[y(O)]-y(O)\right]+6L_{\frac{1}{4}}[y(O)] = \frac{1}{s^{\frac{2}{4}}-1}$$

$$[y(0)] = \frac{1}{(s^{\frac{2}{4}}-1)(s^{\frac{2}{4}}-5s^{\frac{1}{4}}+6)}+\frac{y(0)(s^{\frac{1}{4}}-5)}{(s^{\frac{2}{4}}-5s^{\frac{1}{4}}+6)}+\frac{y^{\frac{1}{4}}(0)}{(s^{\frac{2}{4}}-5s^{\frac{1}{4}}+6)}$$

$$y(0) = L_{\frac{1}{4}}^{-1}\left[\frac{1}{\left(s^{\frac{2}{4}}-1\right)\left(s^{\frac{2}{4}}-5s^{\frac{1}{4}}+6\right)}\right]+y(0)L_{\frac{1}{4}}^{-1}\left[\frac{y(0)\left(s^{\frac{1}{4}}-5\right)}{\left(s^{\frac{2}{4}}-5s^{\frac{1}{4}}+6\right)}\right]+y^{\frac{1}{4}}(0)L_{\frac{1}{4}}^{-1}\left[\frac{1}{\left(s^{\frac{2}{4}}-5s^{\frac{1}{4}}+6\right)}\right]$$

$$= \frac{5}{24}L_{\frac{1}{4}}^{-1}\left[\frac{s^{\frac{1}{4}}}{s^{\frac{2}{4}}-1}\right]+\frac{7}{24}L_{\frac{1}{4}}^{-1}\left[\frac{1}{s^{\frac{2}{4}}-1}\right]+\frac{1}{8}L_{\frac{1}{4}}^{-1}\left[\frac{1}{s^{\frac{1}{4}}-3}\right]$$

$$-\frac{1}{3}L_{\frac{1}{4}}^{-1}\left[\frac{1}{s^{\frac{1}{4}}-2}\right]y(0)-2L_{\frac{1}{4}}^{-1}\left[\frac{1}{s^{\frac{1}{4}}-3}\right]+3L_{\frac{1}{4}}^{-1}\left[\frac{1}{s^{\frac{1}{4}}-2}\right]y(0)$$

$$+L_{\frac{1}{4}}^{-1}\left[\frac{1}{s^{\frac{1}{4}}-3}\right]y^{\frac{1}{4}}(0)-L_{\frac{1}{4}}^{-1}\left[\frac{1}{s^{\frac{1}{4}}-2}\right]\ y^{\frac{1}{4}}(0)$$

$$= \frac{5}{24}\cosh_{\frac{1}{4}}\left(t^{\frac{1}{4}}\right)+\frac{7}{24}\sinh_{\frac{1}{4}}\left(t^{\frac{1}{4}}\right)+\frac{1}{8}E_{\frac{1}{4}}\left(3t^{\frac{1}{4}}\right)-\frac{1}{3}E_{\frac{1}{4}}\left(2t^{\frac{1}{4}}\right)-\ 2E_{\frac{1}{4}}(3t^{\frac{1}{4}})y(0)$$

$$+3E_{\frac{1}{4}}\left(2t^{\frac{1}{4}}\right)y(0)+E_{\frac{1}{4}}\left(3t^{\frac{1}{4}}\right)\ y^{\frac{1}{4}}(0)-E_{\frac{1}{4}}\left(2t^{\frac{1}{4}}\right)\ y^{\frac{1}{4}}(0)$$

$$y(0) = \frac{-23}{8}E_{\frac{1}{4}}\left(3t^{\frac{1}{4}}\right) + \frac{5}{3}E_{\frac{1}{4}}\left(2t^{\frac{1}{4}}\right) + \frac{5}{24}cosh_{\frac{1}{4}}\left(t^{\frac{1}{4}}\right) + \frac{7}{24}sinh_{\frac{1}{4}}\left(t^{\frac{1}{4}}\right)$$

□

Example 3.4. *We solve the following homogeneous FDE using Fractional Laplace Transform,*

$$\left(D^{2\left(\frac{1}{4}\right)} - 4D^{\frac{1}{4}} + 13\right)y(t) = cosh_{\frac{1}{4}}\left(2t^{\frac{1}{4}}\right) \text{ where } y(0) = 1, y^{\frac{1}{4}}(0) = 1$$

Solution: The equation can be written in the form

$$y^{2\left(\frac{1}{4}\right)} - 4y^{\left(\frac{1}{4}\right)} + 13y = cosh_{\frac{1}{4}}2\left(t^{\frac{1}{4}}\right)$$

Applying fractional Laplace transforms to both sides, we have

$$L_{\frac{1}{4}}\left[y^{2\left(\frac{1}{4}\right)}(0) - 4y^{\frac{1}{4}}(0) + 13y(0)\right] = L_{\frac{1}{4}}\left[cosh\ \tfrac{1}{4}\left(2t^{\frac{1}{4}}\right)\right]$$

$$L_{\frac{1}{4}}[y^{2\left(\frac{1}{4}\right)}(0)] - 4L_{\frac{1}{4}}[y^{\frac{1}{4}}(0)] + 13L_{\frac{1}{4}}[y(O)]\ s^{\frac{2}{4}}L_{\frac{1}{4}}[y(0)] - s^{\frac{1}{4}}y(0) - y^{\frac{1}{4}}(0)$$

$$-4[s^{\frac{1}{4}}L_{\frac{1}{4}}[y(O)] - y(O)] + 13L_{\frac{1}{4}}[y(O)] = \frac{s^{\frac{1}{4}}}{s^{\frac{2}{4}}-4}$$

$$L[y(0)] = \frac{s^{\frac{1}{4}}}{(s^{\frac{2}{4}} - 4)(s^{\frac{2}{4}} - 4s^{\frac{1}{4}} + 13)} + \frac{y(0)(s^{\frac{1}{4}} - 4)}{(s^{\frac{2}{4}} - 4s^{\frac{1}{4}} + 13)} + \frac{y^{\frac{1}{4}}(0)}{(s^{\frac{2}{4}} - 4s^{\frac{1}{4}} + 13)}$$

$$y(0) = L_{\frac{1}{4}}^{-1}\left[\frac{s^{\frac{1}{4}}}{\left(s^{\frac{2}{4}} - 4\right)\left(s^{\frac{2}{4}} - 4s^{\frac{1}{4}} + 13\right)}\right] + y(0)L_{\frac{1}{4}}^{-1}\left[\frac{y(0)\left(s^{\frac{1}{4}} - 4\right)}{\left(s^{\frac{2}{4}} - 4s^{\frac{1}{4}} + 13\right)}\right] + y^{\frac{1}{4}}(0)L_{\frac{1}{4}}^{-1}\left[\frac{1}{\left(s^{\frac{2}{4}} - 4s^{\frac{1}{4}} + 13\right)}\right]$$

$$y(0) = \frac{208}{225}E_{\frac{1}{4}}\left(2t^{\frac{1}{4}}\right)\ cosh_{\frac{1}{4}}\left(3t^{\frac{1}{4}}\right) - \frac{208}{675}E_{\frac{1}{4}}\left(2t^{\frac{1}{4}}\right)\ sinh_{\frac{1}{4}}\left(3t^{\frac{1}{4}}\right) + \frac{1}{225}\left[17cosh_{\frac{1}{4}}\left(2t^{\frac{1}{4}}\right) + 8sinh_{\frac{1}{4}}\left(2t^{\frac{1}{4}}\right)\right]$$

□

IV. Conclusion

The Laplace transformation method has been successfully applied to find an exact solution of Fractional Differential Equation. Some results are derived with the proofs. We conclude that the Laplace transformation method is a powerful efficient tool for finding a solution of Fractional Differential Equation.

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