Formulating and Solving a Linear Programming Model for Product- Mix Linear Problems with n Products

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Abstract - This paper demonstrates the use of linear programming methods as applicable in the manufacturing industry. Industrial development strategy is characterized by the efficient use of resources at every production stages. A quantitative decision making tool called linear programming can be used for the optimization problem of product mix. The industry manufacturing firm's profit mainly depends on the proper allocation and usage of available production time, materials (machines), each unit of products P_1 , P_2 , P_3 ,…, P_n and profit per unit for each product of the industry. The linear programming model is used to analyses the linear problem and an optimum solution is reached as well as relevant recommendations to the management of the industry.

Key words: Linear programming, product mix, simplex method, optimization.

1. INTRODUCTION

Linear programming is a mathematical programming technique to optimize performance (example of profit and cost) under a set of resource constraints (machine-hours, man-hours, money, material etc) as specified by an organization.
The paper concerned for product-mix problem that is the one application of the linear programming problems (LPP). This pertains to determining the level of production activities to be carried out during a pre-decided time frame so as to gain the maximum profit. Since the requirements of inputs for the different production process varies, their profitability also differs [1].
Linear programming can be defined as a mathematical technique for determining the best collection of a firm's of limited resources to achieve optimum goal. It is also a mathematical technique used in operational research or management science to solve specific type of problems such as product mix, allocation, transportation and management problems that permits a choice or choices between alternative courses of action [2].
Linear programming being the most prominent operational research technique, it is designed for models with linear objective and constraints functions. A linear programming model can be designed and solve to determine the best courses of action as in the product mix subject to the available constraints [3]. Product mix determination is essential to the success of the industry for a number of important reasons. The manufacturing companies must adopt operations research techniques to enhance best resource utilization that would result in optimal product mix and total profit [3,4].
Thus, this paper focuses on product mix determination based on efficient resource utilization for the machine activity company with respect to time availability and with the consideration of unit product per unit price.

Basic Assumptions of Linear Programming Model

Components of a linear programming model have been identified by as follows:[3,8]

- Proportionality: The level of activity is proportional to the contribution as well as consumption of resources.
- Additivity: Activities contribution and consumption are additive.
- Non-negativity: The values of the activities cannot be negative.
- Linearity: There exist linear relationships between the output of each product and the total quantity of each resource consumed.
- Single or one Objective function: There can be only one objective in a particular problem, either to maximize profit or to minimize cost not both.
- Certainty/ Deterministic: This presupposes that all values and quantities are known. vii. Fixed external factors: This implies that the external environment is assumed not to vary.

Basic Components of a Linear Programming model

A linear programming usually is expressed in inequalities, below are the various component that make up an LP mode

a. Decision variables
b. Objective function
c. Constraints/limitations
d. Non – negativity constraint.

Model Formulation

A linear programming problem consists of the following parameters:

- Decision variables that are mathematical symbols representing level of activity of an operation.
- The objective function that is a linear mathematical relationship describing an objective of the firm in terms of decision variable, that is to be maximized or minimized.
- Constraints that are restrictions placed on the firm.
- Parameters /cost coefficients that are numerical coefficients and constants used in the objective function and constraints equations [4].

Linear programming model formulation steps

Linear programming problems are a collection of the objective function, the set of constraints and the set of non negative constraints. So we consider the steps involved in the mathematical formulation of problems.

Step1: Clearly define the decision variables of the problem $X = (x_1, x_2, x_3, \dots., x_n)$

Step2: Write the objective function as a linear combination of the decision variable,

$$Z = f(x)$$

Step3: Formulate the constraints of the problem as a linear combination of the decision variables [4].

2. MAIN RESULT

Developing and formulating a model for product-mix linear programming problems:

A company manufactures n types of products $P_1, P_2, P_3, \dots. P_n$. Each product uses activities of machines A, B, C. The processing time per unit of $P_1, P_2, P_3, \dots. P_n$ on the activity of machine A are t_1, t_2, t_3, -----t_n hours respectively, on the activity of machine B are $T_1, T_2, T_3, \dots. T_n$hours respectively and on the activity of machine C are $K_1, K_2, K_3, \dots. K_n$hours respectively. The maximum number hours available per a week on the machines A, B and C are h_1 hours, h_2 hours and h_3 hours respectively. Also the profit per unit of selling products $P_1, P_2, P_3, \dots. P_n$ are $b_1, b_2, b_3, \dots. b_m$ in birr $(n = m)$respectively.

Now to develop a linear programming model to determine the product-mix of each of the products such that the total profit maximized:

First identify the management decision problems of the real for the product- mix problem.

Machines	Machine hours/ unit product					Machine hours available per week
	P_1	P_2	P_3	....	P_n	
A	t_1	t_2	t_3		t_n	h_1
B	T_1	T_2	T_3	....	T_n	h_2
C	K_1	K_2	K_3	....	K_n	h_3
Profit or Cost/unit(birr)	b_1	b_2	b_3	....	b_m	

Here to develop and formulate a linear programming model for the real problem of the product volume such that the profit is maximized subject to the availability of constraints that is machine hours.

Let n be the number of products to be manufactured and m be the number of profit or cost (unit/birr) for the products. Again assume that $\mathrm{X_1, X_2, X_3 \dots. X_n}$ be production volume of the products $P_1, P_2, P_3 \dots. P_n$ respectively. The objective function will be either a maximization type or minimization type. The benefit related objective function will come under maximization type whereas the cost related objective function will come under minimization type.

So, the general format of the objective function is represented as:

Maximize or Minimize: $Z = C_1X_1 + C_2X_2 + C_3X_3 + \cdots + C_nX_n$

Having that the amount of resources that required for the activity a_{ij} (say), where a_{ij} is the amount of resources i required for the activity j, i varies from 1 to n and j varies from 1 to m.

So, the generalized format of the technological coefficient is a matrix:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Again say constant b_i is the amount of resources $\boldsymbol{i}$ during the planning period. Here the generalized format of the resource availability matrix is:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

Here the set of constraints is the kind of restrictions on the total amount of a particular resource required to carry out the activities at various levels.

In model, there will be many constraints. Therefore, the constraints functions are:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \quad \leq, \ = \text{or} \quad \geq \qquad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

Hence

$$a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + \cdots + a_{1n}X_n \ \leq, = or \ \geq b-1$$
$$a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + \cdots + a_{2n}X_n \ \leq, = or \ \geq b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{m1}X_1 + a_{m2}X_2 + a_{m3}X_3 + \cdots + a_{mn}X_n \ \leq, = or \ \geq b_m$$

For every decision variable in the linear programming method is non negative variable
Hence, $X_1, X_2, X_3, \ldots X_n \geq 0$
Thus the complete linear programming model of the given product –mix concept is:

Maximize or minimize $Z = C_1X_1 + C_2X_2 + C_3X_3 + \cdots + C_nX_n = \sum_{j=1}^{n} C_jX_j$

Subject to

$$a_{11}X_1 + a_{12}X_2 + a_{13}X_3 + \cdots + a_{1n}X_n \ \leq, = or \ \geq b_1$$
$$a_{21}X_1 + a_{22}X_2 + a_{23}X_3 + \cdots + a_{2n}X_n \ \leq, = or \ \geq b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{m1}X_1 + a_{m2}X_2 + a_{m3}X_3 + \cdots + a_{mn}X_n \ \leq, = or \ \geq b_m$$

$$X_1, X_2, X_3, \ldots X_n \geq 0$$

That is the linear programming problem is one which optimizes (maximizes or minimizes) a linear function subject to a finite collection of linear constraints. Formally, the above Linear programming problem model having n decision variables can be summarized in the following form:

$$\text{Optimize } Z = \sum_{j=1}^{n} C_jX_j$$

Subject to

$$\sum_{i=1}^{m} a_{ij}X_j \ (\leq, =, \geq) b_i, \ i = 1,2,3, \ldots, m$$
$$X_j \geq 0, \qquad j = 1,2,3, \ldots, n$$

Where, C_j, a_{ij}, b_j are constants.

The function, being optimized is referred to as the objective function and the restrictions normally are referred to as constraints. The first m constraints (those with a function of all the variables) are called functional constraints. Similarly the n constraints are called non-negative constraints and the aim is to find the values of the variables X_j.

A feasible solution is a solution for which all the constraints are satisfied and infeasible solution is a solution for which at least one constraint is violated. The problem is to find the values of the decision variableX_j that maximize the objective function Z subject to the m constraints and the non- negativity restriction on the X_j variable. The resulting set of decision variable that maximize the objective function is called optimal solution [4].

3. SIMPLEX –METHOD

The Simplex method is a set of mathematical steps for solving a linear programming problem carried out in a table called a simplex tableau. The tableau organizes the model in to a form that makes applying the mathematical steps easier. Simplex method is an algebraic iterative method that proceeds in a systematic way from initial basic feasible solutions and ultimately reaching the optimal basic feasible solution in finite steps [5]. In order to derive solutions for the linear programming problem using Simplex method, the objective function and the constraints must be standardized.

The characteristics of the standard form are:

– All the constraints must be expressed in the form of equations except the non-negativity constraints which remains in the form of inequalities.
– The right hand side of each constraints equation must be non-negative, that is, the b_i's must be non-negative
– The tableau format can be used to solve the linear programming problem using the Simplex method.

Formulation of a model for the Product-mix problem:

A firm manufactures three products A, B and C. The profits from A, B and C are birr 6, birr 4 and birr 8 respectively. The firm has two machines and given below is the required processing time (in minutes) for each machine on each product.

Machines	Products /unit per birr		
	A	B	C
X	8	6	10
Y	4	4	8

Machines X and Y have 4000 and 5000 birr in minutes respectively. The firm must manufacture 200 A's, 400B's and 100C's but not more than 300A's.

The set up a linear programming problem to maximize profit is as follow.

To formulate the linear programming model for the problem objective is to maximize the profit

$Z = birr(\, 6X_1 + 4X_2 + 8X_3)$ Constraints are on the availability of time in minute for various machines. That is,

$$\text{for machine X}: \; 8X_1 + 6X_2 + 10X_3 \le 4000$$
$$\text{for machine Y}: \; 4X_1 + 4X_2 + 8X_3 \le 5000$$
$$\text{and} \;\; 200 \le X_1 \le 300$$
$$X_2 \ge 400 \; and \; X_3 \ge 100$$

Simplex method is a method used to solve linear programming model with a large number of variables having a concepts of slack variables, surplus variables and artificial variables [6].

Product –mix Problem:

Solve a product-mix problem for which the values of X_1 and X_2 that

$$\text{Maximize:} \;\; Z = 3X_1 + 2X_2$$
$$\text{Subject to}: \;\; -X_1 + 2X_2 \le 4$$
$$3X_1 + 2X_2 \le 14$$
$$X_1 - X_2 \le 3$$
$$X_1 \ge 0, \quad X_2 \ge 0$$

The solution process to maximize first introduce the slack variables $S_1, S_2 \; and \; S_3$. Hence

$Maximize$: $Z = 3X_1 + 2X_2 + 0.S_1 + 0.S_2 + 0.S_3$

We get

$$-X_1 + 2X_2 + S_1 + 0.S_2 + 0.S_3 = 4$$
$$3X_1 + 2X_2 + 0.S_1 + S_2 + 0.S_3 = 14$$
$$X_1 - X_2 + 0.S_1 + 0.S_2 + S_3 = 3$$
$$X_1, \quad X_2, \quad S_1, \quad S_2, \quad S_3 \ge 0$$

The standard form of the linear programming problem is :

$Maximize$: $Z = 3X_1 + 2X_2 + 0.S_1 + 0.S_2 + 0.S_3$

Subject to the constraints

$$-X_1 + 2X_2 + S_1 + 0.S_2 + 0.S_3 = 4$$
$$3X_1 + 2X_2 + 0.S_1 + S_2 + 0.S_3 = 14$$
$$X_1 - X_2 + 0.S_1 + 0.S_2 + S_3 = 3$$

So, the initial simplex table is :

Basic Variables	X_1	X_2	S_1	S_2	S_3	RHS	Ratio
S_1	-1	2	1	0	0	4	
S_2	3	2	0	1	0	14	$\frac{14}{3}$
S_3	1	-1	0	0	1	3	3
$Z_J - C_J$	-3	-2	0	0	0	0	

From the table the pivot element is 1 hence the pivot column is the first column and the pivot row is the third row because of the most negative number on that column and least positive ratio respectively.

$$S_1 \longrightarrow S_1 + S_3$$

Basic Variables	X_1	X_2	S_1	S_2	S_3	RHS	Ratio
S_1	0	1	1	0	1	7	7
S_2	0	5	0	1	-3	5	1
X_1	1	-1	0	0	1	3	
$Z_j - C_j$	0	-5	0	0	3	9	

Here again the second column is the pivot column and the second row is also pivot row.

$$S_2 \longrightarrow \frac{S_2}{5}$$

Basic Variables	X_1	X_2	S_1	S_2	S_3	RHS	Ratio
S_1	0	1	1	0	1	7	7
S_2	0	1	0	$1/5$	$-3/5$	1	1
X_1	1	-1	1	0	1	3	
$Z_j - C_j$	0	-5	0	0	3	9	

If the value of $Z_j - C_j$ (contribution of unit price to the profit) are all positive or zero, the current basic feasible solution is optimal. But there values one or more negative choose the variable corresponding to which the value of $Z_j - C_j$ is least i.e most negative.

$$S_1 \longrightarrow S_1 - X_2, \qquad X_1 \longrightarrow X_1 + X_2, \qquad Z_j - C_j \longrightarrow Z_j - C_j + 5X_2$$

Basic Variables	X_1	X_2	S_1	S_2	S_3	RHS	Ratio
S_1	0	0	1	$-1/5$	$8/5$	6	
X_2	0	1	0	$1/5$	$-3/5$	1	1
X_1	1	1	1	$1/5$	$2/5$	4	
$Z_j - C_j$	0	0	0	1	0	14	

Thus the optimal solution or values are:

$X_1 = 4, \quad X_2 = 1, \quad S_1 = 6$ and

Maximum of $Z = 14$

4. SPECIAL CASES

- Some Linear problems have an infinite number of optimal solutions (alternative or multiple optimal solutions).
- Some Linear problems have no feasible solution (infeasible linear problems).
- Some Linear problems are unbounded. There are points in the feasible region with arbitrary large Z values [7].

5. CONCLUSION and RECOMMENDATION

In Industries, the major constraints to optimize (maximize or minimize) their objective is inefficient management use of resources (poor resource utilization management). The primary reason for using linear programming methodology is to ensure that limited resources are utilized to the fullest extent without any waste and that utilization is made in such a way that the outcomes are expected to be the best possible. That is the use of an operational research technique in the production time horizon helps the industry to improve its objective. The production manager should plan the activities of the industry to produce various products with the given resources of raw materials, man-hours, and machine-time for each product must determine how many products and quantities of each product to produce so as to maximize the total profit. So, as to achieve the plan the industry must have the best possible model called linear programming model. Since the objective of any organization is to make the best utilization of the given resources, linear programming provides powerful technique for effective utilization of these given resources under certain well-defined circumstances.

From this point of view, it can be concluded that an industries should use the linear programming model to solve their linear problems to determine their optimal product mix and optimal solution.

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