Quaternion Quasi-Normal Matrices And Their Eigenvalues

[1]J. Rajeswari , [2]Dr. K. Gunasekaran
[1]Ph.D Research Scholar, [2]HOD and Assistant Professor,
[1,2]Ramanujan Research Centre, PG and Research Department of Mathematics,
Government Arts College(Autonomous), Kumbakonam. Tamil Nadu, India.

Abstract - For square quaternion matrices A and B of the same size, commutativity-like relations such as $AB = \pm BA$, $AB = \pm BA^{CT}$, $AB = \pm BA^{T}$, $AB = \pm B^{T}A$, etc., often cause a special structure of A to be reflected in some special structure for B. We study eigenvalue pairing theorems for B when A is quaternion quasi normal (QQN), a class of quaternion matrices that is a natural generalization of the real normal and complex normal matrices. A new canonical form for QQN matrices is an important tool for our development.
AMS Classification: 15A99, 15A04, 15A15, 15A116, 15A48

Key words - Quaternion matrices, Spectrum, Canonical form, Direct sum, Skew- Hermitian, Orthonormal, Jordan block.

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1. Introduction

Any eigenvalues (quaternion eigenvalues) of a real square matrix A come in conjugate pairs, and corresponding eigenvectors can be chosen in conjugate pairs ($Ax = \lambda x$ if and only if $Ax^{C} = \lambda^{C} x^{C}$); real eigenvectors of A can be associated with its real eigenvalues. If A is diagonalizable, it can therefore be diagonalized in a special way: $A = SAS^{-1}$, $A = L \oplus L^{C} \oplus R$ is diagonal, the diagonal entries of L (if any) are in the open upper half partition of four dimension structure, the diagonal entries of R (if any) are real, $S = [Y \quad Y^{C} \quad Z]$ is non-singular, Y has the same number of columns as L and Z is real.

If A is quaternion normal, it can be quaternion unitary diagonalized in the same way: $A = UAU^{CT}$, $A = L \oplus L^{C} \oplus R$ is diagonal, the diagonal entries of L (if any) are in the open upper half partition of four dimension structure, the diagonal entries of R (if any) are real, $U = [Y \quad Y^{C} \quad Z]$ is unitary, Y has the same number of columns as L and Z is real. This canonical form is different from, but equivalent to, the classical real normal form [5, Theorem 2.5.8] and suggests a wide class of generalizations that play a role in the study of eigenvalue pairing theorems that motivated our investigations. We use standard terminology and notation, as in [5,6]. We let be the set of $m \times n$ matrices with entries in **F=** ¡ or H and write $M_n \equiv M_{n\times n}(\mathrm{H})$. The set of eigenvalues (spectrum) of $A \in M_n(\mathrm{H})$ is denoted by $\sigma(A)$.

Two characterizing properties of a quaternion normal matrix A play an essential role in our discussion: (a) A can be quaternion unitarily diagonalized and (b) a nonzero vector x is a right $\lambda -$eigenvector of A ($Ax = \lambda x$ for some scalar λ) if and only if it is a left eigenvector, necessarily with the same eigenvalue ($x^{CT}A = \lambda x^{CT}$). Eigenvectors of a normal matrix associated with distinct eigenvalues are necessarily orthogonal. If A is quaternion normal, then the quaternion orthogonal complement of the span of any collection of eigenvectors is an invariant subspace of A.

2. Quaternion Quasi unitary matrices

Definition 2.1

A matrix $U \in M_n(\mathrm{H})$ is said to be r-quaternion quasi unitary (r-QQU) if U is quaternion unitary, $U = [Y \quad Y^{C} \quad Z]$, $Y \in M_{n\times r}(\mathrm{H})$, and $Z \in M_{n\times n-2r}$(¡) When $r = 0$, then $U = Z$ is real orthogonal; when $2r = n$ then $U = \begin{bmatrix} Y & Y^{C} \end{bmatrix}$. When the value of the parameter r is not relevant, we say that U is QQU. For a given $Y \in M_{n\times r}(\mathrm{H})$ with quaternion orthonormal columns, the columns of Y need not be quaternion orthogonal to those of Y^{C}. A necessary and sufficient condition for $\begin{bmatrix} Y & Y^{C} \end{bmatrix}$ to have quaternion orthonormal columns is that $Y^{CT}Y = I$ and $Y^{T}Y = 0$, that is, Y has quaternion orthonormal columns that are rectangular and isotropic. If $\begin{bmatrix} Y & Y^{C} \end{bmatrix}$ has quaternion orthonormal columns, then no column of Y can be real since each column of Y must be quaternion orthogonal to every column of Y^{C}.

If $\begin{bmatrix} Y & Y^C \end{bmatrix}$ has quaternion orthonomal columns and $n > 2r$, then there is always a quaternion $X \in M_{n\times n-2r}(\mathrm{H})$ such that $[Y \quad Y^C \quad X]$ (and hence also $[Y^C \quad Y \quad X^C]$) is quaternion unitary. However, any such X has an important property: the column spaces of X and X^C are the identical namely, the quaternion orthogonal complement of the column space of $\begin{bmatrix} Y & Y^C \end{bmatrix}$. We say that a subspace spanned by the columns of $X \in M_{n\times m}(\mathrm{H})$ is self-conjugate if it is the same as the column space of X^C.

Lemma 2.2

Let $\hat{X} \in M_{n\times m}(\mathrm{H})$ have rank $m \geq 1$ and suppose that the column space of $\hat{X}$ is self-conjugate. Then there is a real $Z \in M_{n\times m}(\mathrm{H})$ with quaternion orthonormal columns and the same column space as $\hat{X}$. In particular, if $n \geq m > 2r \geq 0$, $Y \in M_{n\times r}(\mathrm{H})$, $\hat{X} \in M_{n\times m-2r}(\mathrm{H})$, $\tilde{Z} \in M_{n\times n-m}(¡\ )$, and $[Y \quad Y^C \ \hat{X} \ \tilde{Z}] \in M_n(\mathrm{H})$ is quaternion unitary, then there exists a $Z \in M_{n\times m-2r}(¡\ )$ such that $[Y \quad Y^C \ Z \quad \tilde{Z}]$ is quaternion unitary.

Proof:

Since $\hat{X}$ has full rank, there is a matrix $X \in M_{n\times m}(\mathrm{H})$ with quaternion orthonormal columns with the same column space as that of $\hat{X}$. Since the column space of $\hat{X}$, and hence of X, is self-conjugate, there exists a nonsingular $W \in M_r(\mathrm{H})$ such that $X^C = XW$. Then, $I = (X^C)^{CT} X^C = W^{CT} X^{CT} XW = W^{CT} W$, so W is quaternion unitary. Moreover, $X = X^C W^C = XWW^C$, so $X(I - WW^C) = 0$. Since X has full column rank, we must have $WW^C = I$, that is, W is quaternion unitary and coninvolutory and hence it is also symmetric [6, Section 6.4].

Let $p(t)$ be a polynomial such that $V \equiv p(W)$ is a square root of W. Then V is quaternion unitary and symmetric, and hence it is also coninvolutory. Moreover, $X^C = XV^2$ so $X^C V^{-1} = X^C V^C = XV \equiv Z$ is real. Since it is obtained from X by a right quaternion unitary transformation, Z has quaternion orthonormal columns and the same column space as X.

Note 2.3

If the assumption that $\hat{X}$ has full rank is omitted in Lemma 2.2, one may still show that its column space has a real quaternion orthonormal basis [6, Theorem 6.4.24]. The following three assertions are easily verified.

Proposition 2.4

Let $U, V \in M_n(\mathrm{H})$ be r-QQU matrices and let $Q \in M_n(\mathrm{H})$ be real quaternion orthogonal. Then

a. $U^T U = U^{CT} U^C = \begin{bmatrix} 0 & I_r \\ I_r & 0 \end{bmatrix} \oplus I_{n-2r}$ is quaternion unitary, symmetric, and coninvolutory,

b. UV^{CT} is real quaternion orthogonal, and

c. QU is r-QQU.

Proof

b. Suppose $U = \begin{bmatrix} Y_1 & Y_1^C & Z_1 \end{bmatrix}$ and $V = \begin{bmatrix} Y_2 & Y_2^C & Z_2 \end{bmatrix}$ with $Y_1, Y_2 \in M_{n\times r}(\mathrm{H})$. Then UV^{CT} is a product of quaternion unitary matrices and hence is quaternion unitary. However, $UV^{CT} = Y_1 Y_2^{CT} + Y_1^C Y_2^T + Z_1 Z_2^T = 2\,\mathrm{Re}(Y_1 Y_2^{CT}) + Z_1 Z_2^T$ is real, so it is real quaternion orthogonal.

3. Quaternion Quasi-Normal matrices

Definition 3.1

A matrix $A \in M_n(\mathrm{H})$ is said to be quaternion quasi-normal (QQN) if (i) A is quaternion normal. (ii) x^C is an eigenvector of A whenever x is, and (iii) the nullspace of A is self-conjugate, that is $Ax = 0$ if and only if $Ax^C = 0$.

Every real quaternion normal matrix is QQN, but so are several other familiar symmetry classes of quaternion normal matrices. If A is QQN and Q is real quaternion orthogonal, it follows immediately from the definition that A^C and QAQ^T are both QQN [1]. The basic structure of the eigenspaces of a QQN matrix is described in the following lemma, which leads directly to a pleasant canonical form.

Lemma 3.2

Suppose λ is a nonzero eigenvalue of a QQN matrix $A \in M_n(\mathrm{H})$, and let the columns of Y be an quaternion orthonormal basis of the λ-eigenspace of A, so that $AY = \lambda Y$. Then there is a nonzero scalar μ such that $\{x \in \mathrm{H}^n : Ax = \lambda x\}$ = $\{x \in \mathrm{H}^n : A^C x = \mu^C x\}$. If $\mu = \lambda$, then the λ-eigenspace of A is self-conjugate and

$AY^C = \lambda Y^C$; if $\mu \neq \lambda$, then $AY^C = \mu Y^C$, the columns of Y^C are an quaternion orthonormal basis for the μ^C-eigenspace of A^C and $\begin{bmatrix} Y & Y^C \end{bmatrix}$ has quaternion orthonormal columns.

Proof

Let x be a unit λ-eigenvector of A, so there is some scalar μ such that $Ax^C = \mu x^C$ that is, $A^C x = \mu^C x$. Since the conjugate of x^C is not in the nullspace of A, it follows that $\mu \neq 0$. We claim that the λ-eigenspace of A and the μ^C-eigenspace of A^C have the same dimension.

Since A is QQN if and only if A^C is QQN [10], for purposes of obtaining a contradiction it suffices to suppose that the λ-eigenspace of A has dimension greater than that of the μ^C-eigenspace of A^C. Suppose u is a unit vector in the λ-eigenspace of A that is quaternion orthogonal to x, so $Ax = \lambda x,\ Au = \lambda u,\ Ax^C = \lambda x^C$, and there is some scalar v such that $Au^C = vu^C$. But $x + u \neq 0$ and $A(x+u) = \lambda(x+u)$, so $Ax^C + Au^C = A(x+u)^C = \gamma(x+u)^C = \gamma x^C + \gamma u^C$ for some scalar γ. It follows that $\mu = \gamma = v$. Thus, if the columns of Y are a quaternion orthonormal basis of the λ-eigenspace of A (so $AY = \lambda Y$), then $AY^C = \mu Y^C$ and the column space of Y^C is contained in the μ^C-eigenspace of A^C. This shows that the dimension of the μ^C-eigenspace of A^C cannot be less than that of the λ-eigenspace of A, so these two eigenspaces must have the same dimension. Moreover, this argument shows that each eigenspace is the conjugate of the other. If $\lambda = \mu$, the eigenspace is self-conjugate; if $\lambda \neq \mu$ normality of A ensures that the two eigenspaces are quaternion orthogonal.

Theorem 3.3

A matrix $A \in M_n(\mathrm{H})$ is QQN if and only if there is a nonnegative integer r, a r-quaternion quasi-unitary matrix $U = \begin{bmatrix} Y & Y^C & Z \end{bmatrix}$, and a diagonal matrix $A = L_1 \oplus L_2 \oplus L_3$ such that $A = UAU^{CT}$, $L_1, L_2 \in M_r(\mathrm{H})$ are non-singular, and there are nonnegative integers f and g, positive integers $n_1......n_f$, $m_1......m_g$, and $2f + g$ distinct scalars $\lambda_1......\lambda_f$, $\mu_1......\mu_f$, $v_1......v_g$ such that $n_1 + ...+ n_f = r$, $m_1 + ...m_g = n - 2r$, $L_1 = \lambda_1 I_{n_1} \oplus ......\oplus \lambda_f I_{n_f}$, $L_2 = \mu_1 I_{n_1} \oplus ......\oplus \mu_f I_{n_f}$ and $L_3 = v_1 I_{m_1} \oplus ......\oplus v_g I_{m_g}$.

Proof

Suppose A is QQN. Since the nullspace of a QQN matrix is self-conjugate, Lemma 2.2 ensures that if A is singular then there is a real matrix Z with quaternion orthonormal columns that span the nullspace of A. If the column space of Z is all of H^n then $A = Z0Z^T$ and we are done. If not, let λ be any eigenvalue of A acting on the quaternion orthogonal complement of the column space of Z and let the columns of Y be a quaternion orthonormal basis for the λ–eigenspace of A. Lemma 3.2 ensures that either the column space of Y is self-conjugate or there is a nonzero scalar $\mu \neq \lambda$ such that the column space of Y^C is the μ^C–eigenspace of A^C. In the first case, replace Y with a real matrix with quaternion orthonormal columns and the same column space and append it to Z, which then is a real matrix with quaternion orthonormal columns; in the second case, the matrix $\begin{bmatrix} Y & Y^C & Z \end{bmatrix}$ has quaternion orthonormal columns.

If the column space of $\begin{bmatrix} Y & Y^C & Z \end{bmatrix}$ is all of H^n, we are done. If not, proceed in the same way to consider any eigenvalue of A acting on the quaternion orthogonal complement of the column space of $\begin{bmatrix} Y & Y^C & Z \end{bmatrix}$. Augment either Z or Y and Y^C as before and continue until this process exhausts the finitely many distinct eigenvalues of A. At each stage, the construction ensures that any new eigenvalue considered is distinct from any eigenvalue of A previously encountered, so we obtain a QQN matrix that diagonalizes A and gives a representation of the asserted form.

Conversely, suppose that A has a representation of the asserted form. Any eigenvector x of A is in one and only one eigenspace of A, which is spanned by a set of contiguous columns of U corresponding to a unique diagonal block in A. But the span of each such set of contiguous columns is either self-conjugate (the nullspace of A is of this type), or is the conjugate of an eigenspace of A corresponding to a different eigenvalue. In either event, the conjugate of x is an eigenvector of A.

Note 3.4

QQN matrices have polar-type decompositions of all three classical types in which the factors commute.

Theorem 3.5

Let $A \in M_n(\mathrm{H})$ be QQN. Then

a. A commutes with A^{CT} and $A = PV = VP$ with P positive semidefinite and V quaternion unitary.

b. A commutes with A^T and $A = QS = SQ$ with Q quaternion orthogonal and S symmetric.

c. A commutes with A^C (that is, AA^C is real) and $A = RE = ER$ with R real and E coninvolutory.

Proof

Let $A = UAU^{CT}$ be QQN, with $A = L_1 \oplus L_2 \oplus L_3$ and a conformal QQU matrix U. For any given nonzero complex number z, we write $z = re^{i\theta}$ for a unique $r > 0$ and a unique $\theta \in [0, 2\pi)$; we represent $z = 0$ with $r = 0$ and $\theta = 0$ and write $0 = 0e^{i0}$. For any given diagonal matrix $D = diag(d_1......d_p) = diag(r_1e^{i\theta_1}......r_pe^{i\theta_p})$ we define $D^{1/2} \equiv diag(+\sqrt{r_1}e^{i\theta_{1/2}}......+\sqrt{r_p}e^{i\theta_{p/2}})$, $|D| \equiv diag(r_1......r_p)$, and $\Theta(D) \equiv diag(e^{i\theta_1}......e^{i\theta_p})$. The following factors give the asserted decompositions of A:

(a) $P = U(|L_1| \oplus |L_2| \oplus |L_3|)U^{CT}$ and $V = U(\Theta(L_1) \oplus \Theta(L_2) \oplus \Theta(L_3))U^{CT}$

(b) $Q = U(L_2^{-1/2}L_1^{1/2} \oplus L_1^{-1/2}L_2^{1/2} \oplus I)U^{CT}$ and $S = U(L_2^{1/2}L_1^{1/2} \oplus L_2^{1/2}L_1^{1/2} \oplus L_3)U^{CT}$, and

(c) $R = U((L_2^C)^{1/2}L_1^{1/2} \oplus L_2^{1/2}(L_1^C)^{1/2} \oplus |L_3|)U^{CT}$ and

$E = U((L_2^C)^{-1/2}L_1^{1/2} \oplus L_2^{1/2}(L_1^C)^{-1/2} \oplus \Theta(L_3))U^{CT}$

Note 3.6

Finally, we observe that quaternion normal matrices in all of the familiar symmetry classes are QQN.

Theorem 3.7

Let $A \in M_n(\mathrm{H})$ be quaternion normal. In each of the following cases, A is QQN, U is r-QQU, $A = UAU^{CT}$, $A = L_1 \oplus L_2 \oplus L_3$, and the direct summands L_s can be chosen to have the indicated pattern of eigenvalues:

i) A is real $(A^C = A)$: the diagonal entries of L_1 lie in the open upper half plane, $L_2 = L_1^C$ and the diagonal entries of L_3 are real.

ii) A is skew-symmetric $(A^T = -A)$: the diagonal entries of L_1 are either positive or lie in the open upper half plane, $L_2 = -L_1$ and $L_3 = 0$.

iii) A is coninvolutory $(A^C = A^{-1})$: the diagonal entries of L_1 lie in the open exterior of the unit disc, $L_2 = (L_1^C)^{-1}$ and the diagonal entries of L_3 have modulus one.

iv) A is quaternion orthogonal $(A^T = A^{-1})$: the diagonal entries of L_1 lie in the open exterior of the unit disc together with the open circular arc $\{e^{i\theta} : 0 < \theta < \pi\}$, $L_2 = L_1^{-1}$ and $L_3 = I_{m_1} \oplus -I_{m_2}$.

v) A is skew- quaternion orthogonal $(A^T = -A^{-1})$: the diagonal entries of L_1 lie in the open exterior of the unit disc together with the open circular arc $\{e^{i\theta} : \pi/2 < \theta < 3\pi/2\}$, $L_2 = -L_1^{-1}$ and $L_3 = iI_{m_1} \oplus -iI_{m_2}$.

vi) A is pure imaginary $(A^C = -A)$: the diagonal entries of L_1 lie in the open left half plane, $L_2 = -L_1^C$ and the diagonal entries of L_3 are pure imaginary.

vii) A is skew-coninvolutory $(A^C = -A^{-1})$: the diagonal entries of L_1 lie in the open exterior of the unit disc, $L_2 = -(L_1^C)^{-1}$ and the diagonal entries of L_3 have modulus one.

viii) A is symmetric $(A^T = A)$: $r = 0$, L_3 is a diagonal matrix with no restrictions on its entries, and $U = Z$ is real quaternion orthogonal.

Proof

In each of the following cases, let x be a unit vector such that $Ax = \lambda x$. In order to show that A is QQN, it suffices to show that x^C is an eigenvector of A.

i) $Ax^C = (Ax)^C = (\lambda x)^C = \lambda^C x^C$, so x^C is an eigenvector corresponding to the eigenvalue λ^C.

ii) $x^T A = (-Ax)^T = (-\lambda x)^T = -\lambda x^T$, so x^C is an eigenvector corresponding to the eigenvalue $-\lambda$.

iii) $x = A^C Ax = \lambda A^C x$, so $Ax^C = (\lambda^C)^{-1} x^C$; x^C is an eigenvector corresponding to the eigenvalue $(\lambda^C)^{-1}$.

iv) $x = A^T Ax = \lambda A^T x$, so $x^T A = \lambda^{-1} x^T$ and hence $Ax^C = \lambda^{-1} x^C$; x^C is an eigenvector corresponding to the eigenvalue λ^{-1}.

v) $-x = A^T Ax = \lambda A^T x$, so $x^T A = -\lambda^{-1} x^T$ and hence $Ax^C = -\lambda^{-1} x^C$; x^C is an eigenvector corresponding to the eigenvalue $-\lambda^{-1}$.

vi) iA is real.

vii) iA is coninvolutory.

viii) $x^T A = (Ax)^T = (\lambda x)^T = \lambda x^T$, so x^C is an eigenvector corresponding to the eigenvalue λ.

Note 3.8

In the canonical form described in Theorem 3.3, the nonnegative integer $2r$ is the sum of the dimensions of the isotropic eigenspaces of A, or, equivalently, $n-2r$ is the sum of the dimensions of the self-conjugate eigenspaces of A. Thus, r is uniquely determined by A. We say that a QQN matrix A is degenerate if $r = 0$, which Theorem 3.7 (viii) ensures is the case if and only if A is symmetric; otherwise, we say that A is nondegenerate.

4. Eigenvalue pairing theorems

Throughout this section, $A = UAU^{CT} \in M_n(\mathrm{H})$ is a nondegenerate QQN, factored as in Theorem 3.3 with $A = L_1 \oplus L_2 \oplus L_3$ and $U = \begin{bmatrix} Y & Y^C & Z \end{bmatrix}$; let $B \in M_n(\mathrm{H})$ be given. If A is in one of the seven nondegenerate classes enumerated in Theorem 3.7 (i)-(viii), we assume without loss of generality that its eigenvalues have been ordered to achieve the locations stated there for the diagonal entries of L_1, L_2 and L_3.

To illustrate the realm of results we wish to study, consider the prototype case of ordinary commutativity: $AB = BA$. Then $UAU^{CT}B = BUAU^{CT}$, so

$$A(U^{CT}BU) = (U^{CT}BU)A \qquad \text{......(1)}$$

Let

$$U^{CT}BU = \begin{bmatrix} Y^{CT}BY & Y^{CT}BY^C & Y^{CT}BZ \\ Y^T BY & Y^T BY^C & Y^T BZ \\ Z^T BY & Z^T BY^C & Z^T BZ \end{bmatrix} \equiv \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \qquad \text{.......(2)}$$

$$V \equiv \begin{bmatrix} Y & Y^C \end{bmatrix} \in M_{n\times 2r}(\mathrm{H}),$$

$$C \equiv V^{CT}BV = \begin{bmatrix} Y^{CT}BY & Y^{CT}BY^C \\ Y^TBY & Y^TBY^C \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \in M_{2r}(\mathrm{H}) \qquad ......(3)$$

and $\Delta \equiv L_1 \oplus L_2 \in M_{2r}(\mathrm{H})$. Writing out Eq.1 in block form gives the identity

$$\begin{bmatrix} \Delta C & \Delta(V^{CT}BZ) \\ L_3(Z^TBV) & L_3B_{33} \end{bmatrix} = \begin{bmatrix} C\Delta & (V^{CT}BZ)L_3 \\ (Z^TBV)\Delta & B_{33}L_3 \end{bmatrix} \qquad ....(4)$$

Because L_1, L_2 and L_3 (and hence also Δ and L_3) have pairwise disjoint spectra, Sylvester's Theorem[6, Theorem 4.4.6] and equality of the (1,2) blocks in Eq.4, as well as the (2,1) blocks, implies that $V^{CT}BZ = 0$ and $Z^TBV = 0$. Thus,

$$U^{CT}BU = \begin{bmatrix} C & V^{CT}BZ \\ Z^TBV & Z^TBZ \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & B_{33} \end{bmatrix},$$

is block diagonal and unitarily similar to B; the column spaces of $V = \begin{bmatrix} Y & Y^C \end{bmatrix}$ and Z are each invariant under B. We are interested in the eigenstructure of C, which is the restriction of B to the column space of $\begin{bmatrix} Y & Y^C \end{bmatrix}$.

Writing out the equation $\Delta C = C\Delta$ from the (1,1) blocks of Eq.4 gives the identity

$$\begin{bmatrix} L_1B_{11} & L_1B_{12} \\ L_2B_{21} & L_2B_{22} \end{bmatrix} = \begin{bmatrix} B_{11}L_1 & B_{12}L_2 \\ B_{21}L_1 & B_{22}L_2 \end{bmatrix} \qquad ....(5)$$

which tells us that $B_{12} = B_{21} = 0$ since $\sigma(L_1) \cap \sigma(L_2) = \phi$. Thus, the column spaces of Y and Y^C are each invariant under $B, C = Y^{CT}BY \oplus Y^TBY^C$, and $U^{CT}BU = Y^{CT}BY \oplus Y^TBY^C \oplus Z^TBZ$. Although there is nothing special about the eigenstructure of a quaternion symmetric matrix, in this case we get something interesting if we assume that B is symmetric: $Y^{CT}BY = (Y^TBY^C)^T$ is similar to Y^TBY^C, so every block in the Jordan canonical form of C appears an even number of times. In particular, every eigenvalue of C has even multiplicity.

Similar calculations, some with the help of Proposition 2.4, show that other commutativity-related assumptions about AB have useful consequences for ΔC: If $AB = \pm BA$, then $\Delta C = \pm C\Delta$; if $AB = \pm BA^{CT}$, then $\Delta C = \pm C\Delta^C$; and if $AB = \pm BA^T$, then $\Delta C = \pm C(L_2 \oplus L_1)$. Under natural conditions on the spectra of L_1 and L_2, these relations imply that C is block diagonal. Moreover, certain conditions on B ensure various pairings of the eigenvalues of C.

Other authors have considered implications of the condition $AB = BA^T$ for various classes of matrices (see [2,4,8,9]). The following results have their origin in a study of the spectral properties of quaternion unitary operators induced by ergodic measure preserving transformations[1,3].

Theorem 4.1

Let $A = UAU^{CT} \in M_n(\mathrm{H})$ be a nondegenerate QQN, factored as in Theorem 3.3 with $A = L_1 \oplus L_2 \oplus L_3$ and $U = \begin{bmatrix} Y & Y^C & Z \end{bmatrix}$. Let $B \in M_n(\mathrm{H})$ be given, and set $C = V^{CT}BV$ with $V = \begin{bmatrix} Y & Y^C \end{bmatrix}$.

Suppose that any one of the following conditions is satisfied:

i) $AB = BA$,
ii) $AB = -BA$ and $\sigma(L_1) \cap \sigma(-L_2) = \phi$,

iii) $AB = BA^{CT}$ and $\sigma(L_1) \cap \sigma(L_2^C) = \phi$

iv) $AB = -BA^{CT}$ and $\sigma(L_1) \cap \sigma(-L_2^C) = \phi$, or

v) $AB = -BA^T$, $\sigma(L_1) \cap \sigma(-L_1) = \phi$, and $\sigma(L_2) \cap \sigma(-L_2) = \phi$.

Then $C = Y^{CT}BY \oplus Y^T BY^C$ is block diagonal. Moreover,

a) If $B^T = B$, then each block in the Jordan canonical form of C occurs an even number of times, so every eigenvalue of C has even multiplicity.

b) If B is real and $J_m(\lambda)$ is a Jordan block of C, then so is $J_m(\lambda^C)$. Each Jordan block of C corresponding to a real eigenvalue occurs an even number of times, so each real eigenvalue of C (if any) has even multiplicity.

c) If $B^T = -B$ and $J_m(\lambda)$ is a Jordan block of C, then so is $J_m(-\lambda)$. Any nilpotent Jordan block of C occurs an even number of times, so if C is singular, zero is an eigenvalue with even multiplicity.

d) If $B^C = -B$ and if $J_m(\lambda)$ is a Jordan block of C, then so is $J_m(-\lambda^C)$. Each Jordan block of C corresponding to a pure imaginary eigenvalue occurs an even number of times, so each pure imaginary eigenvalue of C (if any) has even multiplicity.

Proof

Under each of the assumptions (i)-(v), inspection of the analog of Eq.5 in each case shows that the off-diagonal blocks B_{12} and B_{21} are zero.

a) If B is symmetric, then $B_{22} = B_{11}^T$, so $C = B_{11} \oplus B_{11}^T$. The Jordan canonical form of a square quaternion matrix and its transpose are the same, so every block in the Jordan canonical form of C occurs an even number of times.

b) If B is real, then $B_{22} = (Y^{CT}BY)^C = B_{11}^C$, so the Jordan blocks of C occur in conjugate pairs of the form $J_m(\lambda) \oplus J_m(\lambda^C)$. If λ is real, then its Jordan blocks occur in pairs of the form $J_m(\lambda) \oplus J_m(\lambda)$, so each real eigenvalue has even multiplicity.

c) If B is skew-symmetric, then $B_{22} = -B_{11}^T$, so the Jordan blocks of C occur in pairs of the form $J_m(\lambda) \oplus J_m(-\lambda)$. If C is singular, then its nilpotent Jordan blocks occur in pairs of the form $J_m(0) \oplus J_m(0)$, so zero is an eigenvalue with even multiplicity.

d) If B is pure imaginary, then $B = iD$ for some real $D \in M_n(\mathbb{H})$ and the assertions follow from (b).

Note 4.2

Of course, the eigenvalues of C need not be eigenvalues of B. However, certain additional conditions ensure that $V^{CT}BZ$ and $Z^T BV$ are both zero. B_{13}, B_{23}, B_{31} and B_{32} are all zero, $U^{CT}BU$ is block diagonal, and the eigenvalues of C are also eigenvalues of B.

Theorem 4.3

Let $A = UAU^{CT} \in M_n(\mathbb{H})$ be a nondegenerate QQN, factored as in Theorem 3.3, with $A = L_1 \oplus L_2 \oplus L_3$ and $U = \begin{bmatrix} Y & Y^C & Z \end{bmatrix}$. Let $B \in M_n(\mathbb{H})$ be given, and set $C = V^{CT}BV$ with $V = \begin{bmatrix} Y & Y^C \end{bmatrix}$.

Suppose that any one of the following conditions is satisfied:

i) $AB = BA$,

ii) $AB = -BA$, $\sigma(L_1) \cap \sigma(-L_2) = \phi$, $\sigma(L_1) \cap \sigma(-L_3) = \phi$, and $\sigma(L_2) \cap \sigma(-L_3) = \phi$ (these conditions are satisfied if A is coninvolutory or skew-coninvolutory),

iii) $AB = BA^{CT}$, $\sigma(L_1) \cap \sigma(L_2^C) = \phi$, $\sigma(L_1) \cap \sigma(L_3^C) = \phi$, and $\sigma(L_2) \cap \sigma(L_3^C) = \phi$ (these conditions are satisfied if A is skew-quaternion orthogonal, coninvolutory, or skew-coninvolutory).

iv) $AB = -BA^{CT}$, $\sigma(L_1) \cap \sigma(-L_2^C) = \phi$, $\sigma(L_1) \cap \sigma(-L_3^C) = \phi$, and $\sigma(L_2) \cap \sigma(-L_3^C) = \phi$ (these conditions are satisfied if A is quaternion orthogonal, coninvolutory, or skew-coninvolutory), or

v) $AB = -BA^T$, $\sigma(L_1) \cap \sigma(-L_1) = \phi$, $\sigma(L_1) \cap \sigma(-L_3) = \phi$, $\sigma(L_2) \cap \sigma(-L_2) = \phi$, and $\sigma(L_2) \cap \sigma(-L_3) = \phi$ (these conditions are satisfied if A is real, pure imaginary, or skew-symmetric).

Then $U^{CT}BU = Y^{CT}BY \oplus Y^T BY^C \oplus Z^T BZ = C \oplus Z^T BZ$, every eigenvalue of C is an eigenvalue of B, and C satisfies each of the conclusions (a)-(d) of Theorem 4.1.

Certain conditions force the diagonal blocks B_{11} and B_{22} to be zero, and certain conditions on B ensure that the eigenvalues of C are paired.

Theorem 4.4

Let $A = UAU^{CT} \in M_n(H)$ be a nondegenerate QQN, factored as in Theorem 3.3, with $A = L_1 \oplus L_2 \oplus L_3$ and $U = \begin{bmatrix} Y & Y^C & Z \end{bmatrix}$. Let $B \in M_n(H)$ be given, and set $C = V^{CT}BV$ with $V = \begin{bmatrix} Y & Y^C \end{bmatrix}$.

Suppose that any one of the following conditions is satisfied:

i) $AB = -BA$, $\sigma(L_1) \cap \sigma(-L_1) = \phi$, and $\sigma(L_2) \cap \sigma(-L_2) = \phi$,

ii) $AB = BA^{CT}$, $\sigma(L_1) \cap \sigma(L_1^C) = \phi$, and $\sigma(L_2) \cap \sigma(L_2^C) = \phi$,

iii) $AB = -BA^{CT}$, $\sigma(L_1) \cap \sigma(-L_1^C) = \phi$, and $\sigma(L_2) \cap \sigma(-L_2^C) = \phi$,

iv) $AB = BA^T$, or

v) $AB = -BA^T$, and $\sigma(L_1) \cap \sigma(-L_2) = \phi$.

Then $$C = \begin{bmatrix} 0 & Y^{CT}BY^C \\ Y^T BY & 0 \end{bmatrix}$$

Moreover,

a) Every non-singular Jordan block of C^2 occurs an even number of times, and every eigenvalue of C^2 has even multiplicity.

b) If B is either real or pure imaginary, and if $J_m(\lambda)$ is a Jordan block of C, then so are $J_m(-\lambda)$ and $J_m(\pm\lambda^C)$. Thus, the eigenvalues of C occur in $\pm$ conjugate quadruplets with the same multiplicities.

c) If $B^{CT} = B$, then the eigenvalues of C are real and occur in $\pm$ pairs with the same multiplicities. In fact they are singular values of $Y^{CT}BY^C$ together with their negatives.

Proof

Under each of the assumptions (i)-(v), inspection of the analog of Eq.5 in each case shows that $B_{11} = B_{22} = 0$.

a) We have $C^2 = B_{12}B_{21} \oplus B_{21}B_{12}$, and the non-singular Jordan blocks of $B_{12}B_{21}$ and $B_{21}B_{12}$ are always the same; their nilpotent Jordan structures can be different, but zero is an eigenvalue of the same multiplicity for both. Hence, C^2 has an even number of zero eigenvalues.

b) If B is real, then $C = \begin{bmatrix} 0 & B_{12} \\ B_{12}^C & 0 \end{bmatrix}$. Since every square quaternion matrix is consimilar to a real matrix, there is a real R and a non-singular S such that $B_{12} = SR(S^C)^{-1}$. Let $X \equiv S \oplus S^C$ and let $K \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}$, so $K^{-1} = K^T$. A calculation reveals that $(XK^{-1})^{-1}C(XK^{-1}) = -R \oplus R$. Thus, if $J_m(\lambda)$ is a Jordan block of C, then so are $J_m(-\lambda)$, $J_m(\lambda^C)$, and $J_m(-\lambda^C)$.

c) If $B = B^{CT}$, then $C = \begin{bmatrix} 0 & B_{12} \\ B_{12}^{CT} & 0 \end{bmatrix}$. Let $B_{12} = U\Sigma V$ be a singular value decomposition of B_{12}, and set $X \equiv U \oplus V^{CT}$. Then $(XK^{-1})^{-1}C(XK^{-1}) = -\Sigma \oplus \Sigma$ [5, Theorem 7.3.7]

Note 4.5

Certain conditions on L_1, L_2 and L_3 ensure that $U^{CT}BU = C \oplus Z^T BZ$, so the eigenvalues of C are also eigenvalues of B.

Theorem 4.6

Let $A = UAU^{CT} \in M_n(\mathrm{H})$ be a nondegenerate QQN, factored as in Theorem 3.3, with $A = L_1 \oplus L_2 \oplus L_3$ and $U = \begin{bmatrix} Y & Y^C & Z \end{bmatrix}$. Let $B \in M_n(\mathrm{H})$ be given, and set $C = V^{CT}BV$ with $V = \begin{bmatrix} Y & Y^C \end{bmatrix}$.

Suppose that any one of the following conditions is satisfied:

i) $AB = -BA$, $\sigma(L_1) \cap \sigma(-L_1) = \phi$, $\sigma(L_1) \cap \sigma(-L_3) = \phi$, and $\sigma(L_2) \cap \sigma(-L_2) = \phi$, and $\sigma(L_2) \cap \sigma(-L_3) = \phi$ (these conditions are satisfied if A is real, pure imaginary, or skew-symmetric).

ii) $AB = BA^{CT}$, $\sigma(L_1) \cap \sigma(L_1^C) = \phi$, $\sigma(L_2) \cap \sigma(L_2^C) = \phi$, $\sigma(L_1) \cap \sigma(L_3^C) = \phi$, and $\sigma(L_2) \cap \sigma(L_3^C) = \phi$ (these conditions are satisfied if A is real),

iii) $AB = -BA^{CT}$, $\sigma(L_1) \cap \sigma(-L_1^C) = \phi$, and $\sigma(L_2) \cap \sigma(-L_2^C) = \phi$, $\sigma(L_1) \cap \sigma(-L_3^C) = \phi$, and $\sigma(L_2) \cap \sigma(-L_3^C) = \phi$ (these conditions are satisfied if A is pure imaginary),

iv) $AB = BA^T$, or

v) $AB = -BA^T$, $\sigma(L_1) \cap \sigma(-L_2) = \phi$, $\sigma(L_1) \cap \sigma(-L_3) = \phi$, and $\sigma(L_2) \cap \sigma(-L_3) = \phi$ (these conditions are satisfied if A is coninvolutory or skew-coninvolutory),

Then $$U^{CT}BU = \begin{bmatrix} 0 & Y^{CT}BY^C \\ Y^T BY & 0 \end{bmatrix} \oplus Z^T BZ$$

Every eigenvalue of C is an eigenvalue of B, and C satisfies the conclusions (a)-(c) of Theorem 4.4.

Proof

Proceed as before.

Note 4.7

The relations $AB = \pm B^T A$ and $AB^T \pm BA$ also lead to eigenvalue pairings, but via a somewhat different path.

Lemma 4.8

Let $T \in M_{2r}(\mathrm{H})$ be given.

a) If T is skew-quaternion Hamiltonian, then every Jordan block of T occurs an even number of times.

b) Suppose T is quaternion Hamiltonian. Then T^2 is skew-quaternion Hamiltonian. If $J_m(\lambda)$ is a nonsingular Jordan block of T, then so is $J_m(-\lambda)$. Every odd-sized singular Jordan block of T occurs an even number of times, and zero is an eigenvalue of T with even multiplicity.

Proof

a) The asserted pairing follows from the fact that a skew-quaternion Hamiltonian matrix is similar (via a symplectic similarity) to the direct sum of a matrix and its transpose [7, Theorem 6].

b) If T is quaternion Hamiltonian, block multiplication reveals that T^2 is skew-quaternion Hamiltonian and that

$$\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} E & F \\ G & -E^T \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} = \begin{bmatrix} E & F \\ G & -E^T \end{bmatrix}^T,$$

so $-T$ is similar to the transpose of T, and hence to T itself. This observation proves the asserted pairing of the nonsingular Jordan blocks of T and T^2. One checks that $(J_{2m+1}(0))^2$ is similar to $J_m(0) \oplus J_{m+1}(0)$ and that $(J_{2m}(0))^2$ is similar to $J_m(0) \oplus J_m(0)$. If $J_{2m+1}(0)$ is an odd-sized nilpotent Jordan block of T, the fact that T^2 is Skew-Quaternion Hamiltonian means that its Jordan canonical form contains each of the blocks $J_m(0)$ and $J_{m+1}(0)$ an even number of times; their respective parities are unaffected by the presence or absence of $J_{2m}(0)$ in the Jordan form of T. Thus, the Jordan form of T must contain an even number of copies of $J_{2m+1}(0)$.

Theorem 4.9

Let $A \in M_n(\mathrm{H})$ be quaternion normal and let $B \in M_n(\mathrm{H})$ be given.

a) Suppose that A is not quaternion symmetric and that A and B satisfy at least one of the four conditions $AB = B^T A$, $AB = -B^T A$, $AB^T = BA$, or $AB^T = -BA$. Suppose either that A is coninvolutory and factored as in Theorem 3.7(iii), or that A is skew-quaternion coninvolutory and factored as in Theorem 3.7(vii). Let $U^{CT} BU = [B_{st}]$ be defined as in eqn. (2). Then $U^{CT} BU = B_{11} \oplus B_{22} \oplus B_{33}$ is block diagonal. If $AB = B^T A$ or $AB^T = BA$, then every Jordan block of $B_{11} \oplus B_{22}$ occurs with even multiplicity. If $AB = -B^T A$ or $AB^T = -BA$, and if $J_m(\lambda)$ is a Jordan block of $B_{11} \oplus B_{22}$, then so is $J_m(-\lambda)$.

b) Suppose that $A \in M_n(\mathrm{H})$ is nonzero and skew-quaternion symmetric, factored as in Theorem 3.7(ii).

b1) Suppose that $AB = B^T A$ and $BA = AB^T$. Then $U^{CT} BU = C \oplus B_{33}$. Moreover, C is Skew-quaternion Hamiltonian, so every block in the Jordan canonical form of C occurs an even number of times.

b2) Suppose that $AB = -B^T A$ and $BA = -AB^T$. Then $U^{CT} BU = C \oplus B_{33}$. Moreover, C is Quaternion Hamiltonian. If $J_m(\lambda)$ is a nonsingular Jordan block of C, then so is $J_m(-\lambda)$. Every odd-sized singular Jordan block of

C occurs an even number of times, and zero is an eigenvalue of C with even multiplicity. Every Jordan block of C^2 occurs an even number of times.

Proof

a) The assumptions ensure that A is a nondegenerate QQN with $L_2 = \pm(L_1^C)^{-1}$. Moreover, L_3 is non-singular and the diagonal entries of L_1 lie outside the open unit disk. Using the notation and invoking its conclusions, it suffices to observe that $|\lambda_s \lambda_t| > 1 > |\mu_s \mu_t|$ for all $s, t = 1......r$. Moreover, B_{11} is similar to $\pm B_{22}$, which ensures that the assertions about pairings of the Jordan blocks of $B_{11} \oplus B_{22}$ are correct.

b1) Again, A is a nondegenerate QQN. The diagonal entries of L_1 are either positive or in the open upper half plane, and $L_2 = -L_1$. The key observation is that, under these conditions, L_1 is a polynomial in L_1^2, and L_1^2 commutes with B_{11}, it follows that L_1 commutes with B_{11}. Thus, $B_{22}^T = (L_1)^{-1} B_{11} L_1 = B_{11}$. Also, $B_{12}^T = (L_1)^{-1} B_{12} L_2 = -(L_1)^{-1} B_{12} L_1 = -B_{12}$, so B_{12} is skew-quaternion symmetric. A similar computation shows that B_{21} is also skew-quaternion symmetric, so C is skew-Quaternion Hamiltonian.

b2) One can argue as in (b1).

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