© IJEDR 2018 | Volume 6, Issue 2 | ISSN: 2321-9939

Application of Eigenvalues and Eigenvectors to Systems of First Order Differential Equations

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Abstract - This paper provides a method for solving systems of first order ordinary differential equations by using eigenvalues and eigenvectors. Solutions will be obtained through the process of transforming a given matrix into a diagonal matrix.

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1. INTRODUCTION

The first major problem of linear algebra is to understand how to solve the basis linear system Ax = b and what the solution means. We have explored this system from three points of view: from an operational point of view (the mechanics of computing solutions), from the perspective of matrix theory and from the vantage of vector space theory. The second major problem of linear algebra is the eigenvalue problem which is more sophisticated. [9]

Computing eigenvalues boils down to solving a polynomial equation. But determining solutions to polynomial equations can be a formidable task. It was proven in the nineteenth century that it's impossible to express the roots of a general polynomial of degree five or higher using radicals of the coefficients. This means that there does not exist a generalized version of the quadratic formula for polynomials of degree greater than four, and general polynomial equations cannot be solved by a finite number of arithmetic operations involving $+, -, \times$ and $\div$. Unlike solving $A\boldsymbol{X} = b$, the eigenvalue problem generally requires an infinite algorithm, so all practical eigenvalue computations are accomplished by iterative methods.[2]

Many applications of matrices in both engineering and science utilize eigenvalues and, sometimes, eigenvectors. Systems of first order ordinary differential equations arise in many areas of mathematics and engineering. A number of techniques have been developed to solve such systems of equations; for example the Laplace transform. In this paper, we shall use eigenvalues and eigenvectors to obtain the solution.

2. OBJECTIVES

General objective

The general objective of this study is to develop alternative method for solving systems of first order ordinary differential equations.

Specific objective

The specific objective of this study is to solve systems of first order ordinary differential equations.

3. THE EIGENVALUE PROBLEM

3.1. Diagonalization of a Square Matrix

Let A be an $n \times n$ matrix. So, what we really need to know is how the powers of A, say A^k, behave? In general, this is very hard, but here is an easy case we can handle:

What if $A = [a_{ij}]$ is diagonal?

Since we'll make extensive use of diagonal matrices, let's recall a notation that the matrix $diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$ is the $n \times n$ diagonal matrix with entries $\lambda_1, \lambda_2, ..., \lambda_n$ down the diagonal. For example,

$$diag\ \{\lambda_1, \lambda_2, \lambda_3\} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

By matching up the i^{th} row and j^{th} column of A, we see that the only time we could have a nonzero entry in A^2 is when $i = j$, and in that case the entry is a_{ii}^2. A similar argument applies to any power of A. In summary, we have this handy fact.

Theorem 3.1.1*'2*

Let $D = diag\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be a diagonal matrix. Then, $D^k = diag\{\lambda^k{}_1, \lambda^k{}_2, \dots, \lambda^k{}_n\}$, where k is a positive integer.

Let's now consider a 3×3 matrix A. If we could find three linearly independent eigenvectors $\boldsymbol{v_1}$, $\boldsymbol{v_2}$ and $\boldsymbol{v_3}$ corresponding to the eigenvalues λ_1, λ_2 and λ_3, we would have $A\boldsymbol{v_1} = \lambda_1\boldsymbol{v_1}$, $A\boldsymbol{v_2} = \lambda_2\boldsymbol{v_2}$, and $A\boldsymbol{v_3} = \lambda_3\boldsymbol{v_3}$. In matrix form, we have

$$A[\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}] = [\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = [\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}] diag[\lambda_1, \lambda_2, \lambda_3]$$

Now, set $P = [\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}]$ and $D = diag[\lambda_1, \lambda_2, \lambda_3]$. Then, P is invertible since the columns of P are linearly independent. Multiplying both sides of $AP = PD$ by P^{-1} to the left, we get $P^{-1}AP = D$. This is a beautiful equation, because it makes the powers of A simple to understand. The procedure we just went through is reversible as well. In other words, if P is an invertible matrix such that $P^{-1}AP = D$, then we deduce that $AP = PD$ and conclude that the columns of P are linearly independent eigenvectors of A. We make the following definition and follow it with a simple but key theorem relating similar matrices.

Definition 3.1.1

Let A and B are square matrices of the same order. Then, A is said to be similar to B if there exists an invertible matrix P such that $P^{-1}AP = B$. The matrix P is called a similarity transformation matrix.

Note that if A is similar to B, then B is similar to A and the two matrices are called similar matrices.

Definition 3.1.2

The matrix A is diagonalizable if it is similar to a diagonal matrix, that is, there is an invertible matrix P and diagonal matrix D such that $P^{-1}AP = D$. In this case we say that P is a diagonalizing matrix for A or that P diagonalizes A.

We can be more specific about when a matrix is diagonalizable. As a first step, notice that the calculations that we began can easily be written in terms of an $n \times n$ matrix instead of a 3×3 matrix. What these calculations prove is the following basic fact.

Theorem 3.1.2

The $n \times n$ matrix A is diagonalizable if and only if there exists a linearly independent set of eigenvectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_n}$ of A, in which case $P = [\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_n}]$ is a diagonalizing matrix for A.

In general we have the following steps to digonalize a matrix A if possible.

Let A be an $n \times n$ matrix.

1. Find n linearly independent eigenvectors for A (if possible) say, $p_1, p_2, \dots, p_n$. If n linearly independent eigenvectors do not exist, then A is not diagonalizable.
2. From the matrix P having $\boldsymbol{p_1}, \boldsymbol{p_2}, \dots, \boldsymbol{p_n}$ as its column vectors.
3. The matrix $D = P^{-1}AP$ will be diagonal with λ_i as its successive diagonal entries, where λ_i is the eigenvalue corresponding to $\boldsymbol{p_i}$.

Note that the order of the eigenvectors used to form P will determine the order in which the eigenvalues appear on the main diagonal of D.

Example 3.1.1

Apply the results of the preceding discussion to the matrix

$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ or explain why they fail to apply.

Solution

We know that the eigenvalues of a triangular matrix are the elements in the main diagonal. That is,

$\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = 2$ are the eigenvalues of A.

Let's now find the corresponding eigenvectors of A.

For $\lambda_1 = 1$, apply Gauss–Jordan elimination to the matrix $(1I - A)$.

$$(I - A)\boldsymbol{X} = \begin{bmatrix} 1-2 & -1 & -1 \\ 0 & 1-1 & -1 \\ 0 & 0 & 1-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives a general eigenvector of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Hence, the eigenspace corresponding to $\lambda_1 = 1$ has basis $\{(-1, 1, 0)\}$.

Similarly, the eigenspace corresponding to $\lambda_2 = 2$ has basis $\{(1, 0, 0)\}$.

All we could come up with here is two eigenvectors. As a matter of fact, they are linearly independent since one is not a multiple of the other. But, they aren't enough and there is no way to find a third eigenvector, since we have found them all. Therefore, we have no hope of diagonalizing this matrix. The problem is that A is defective, since the algebraic multiplicity of $\lambda_2 = 2$ exceeds the geometric multiplicity of this eigenvalue.

So, it would be very handy to have some working criterion for when we can manufacture linearly independent sets of eigenvectors. The next theorem gives us such a criterion.

Theorem 3.1.3

Let $\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_k}$ be a set of eigenvectors of the matrix A such that corresponding eigenvalues are all distinct. Then, the set of vectors $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_k}\}$ is linearly independent.

Proof

Suppose the set is linearly dependent. Discard redundant vectors until we have a smallest linearly dependent subset such as $\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_m}$ with v_i belonging to λ_i. All the vectors have nonzero coefficients in a linear combination that sums to zero, for we could discard the ones that have zero coefficient in the linear combination and still have a linearly dependent set. So there is some linear combination of the form

$$c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \dots + c_m\boldsymbol{v_m} = \boldsymbol{0} \qquad (1)$$

with each $c_j \neq 0$ and v_j belonging to the eigenvalue λ_j.
Multiply equation 1 by λ_1 to obtain the equation

$$c_1\lambda_1\boldsymbol{v_1} + c_2\lambda_1\boldsymbol{v_2} + \ldots + c_m\lambda_1\boldsymbol{v_m} = \boldsymbol{0} \qquad (2)$$

Next, multiply equation 1 on the left by A to obtain

$$0 = A(c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \ldots + c_m\boldsymbol{v_m}) = c_1A\boldsymbol{v_1} + c_2A\boldsymbol{v_2} + \ldots + c_mA\boldsymbol{v_m}$$

That is,

$$c_1\lambda_1\boldsymbol{v_1} + c_2\lambda_2\boldsymbol{v_2} + \ldots + c_k\lambda_m\boldsymbol{v_m} = \boldsymbol{0} \qquad (3)$$

Now subtract equation 3 from equation 2 to obtain

$$0\boldsymbol{v_1} + c_2(\lambda_1 - \lambda_2)\boldsymbol{v_2} + \ldots + c_k(\lambda_1 - \lambda_m)\boldsymbol{v_m} = \boldsymbol{0}$$

This is a new nontrivial linear combination (since $c_2(\lambda_1 - \lambda_2) \neq 0$) of fewer terms, that contradicts our choice of $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}$. It follows that the original set of vectors must be linearly independent. ■
Here is an application of the Theorem that is useful for many problems.
*Corollary*3.1.1
If the $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.
Proof
We can always find one nonzero eigenvector v_i for each eigenvalue λ_i of A. By the preceding theorem, the set $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}$ is linearly independent.
Thus A is diagonalizable by the diagonalization theorem. ■
Note that just because the $n \times n$ matrix A has fewer than n distinct eigenvalues, you may not conclude that it is not diagonalizable.
A simple example is the identity matrix, which is certainly diagonalizable (it's already diagonal) but has only 1 as an eigenvalue.
3.2. Symmetric Matrices and Diagonalization
For most matrices, you must go through much of the diagonalization process before determining whether diagonalization is possible. One exception is with a triangular matrix that has distinct entries on the main diagonal. Such a matrix can be recognized as diagonalizable by inspection. In this section, we will study another type of matrix that is guaranteed to be diagonalizable: a symmetric matrix.
Theorem 3.2.1
If A is an $n \times n$ symmetric matrix, then the following properties are true.

1. A is diagonalizable.
2. All eigenvalues of A are real.
3. If λ is an eigenvalue of A with algebraic multiplicity k, then λ has k linearly independent eigenvectors. That is, the eigenspace of λ has dimension k.

4. SOLVING SYSTEMS OF FIRST ORDER DIFFERENTIAL EQUATIONS

Consider a system of ordinary first order differential equations of the form

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \qquad\qquad\quad \vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{aligned}$$

Where, $a_{ij} \in \mathbb{R}$.
Now, we shall use eigenvalues and eigenvectors to obtain the solution of this system.

- ✓ Our first step will be to rewrite the system in the matrix form $\boldsymbol{X}' = A\boldsymbol{X}$ where A is the $n \times n$ coefficient matrix of constants, $\boldsymbol{X}$ is the $n \times 1$ column vector of unknown functions and $\boldsymbol{X}'$ is the $n \times 1$ column vector containing the derivatives of the unknowns.
- ✓ The main step will be to use the diagonalizing matrix of A to diagonalise the system. This process will transform $\boldsymbol{X}' = A\boldsymbol{X}$ into the form $\boldsymbol{Y}' = D\boldsymbol{Y}$, where D is a diagonal matrix.
- ✓ Finally, we shall find that this new diagonal system of differential equations can be easily solved. This special solution will allow us to obtain the solution of the original system.

Note that in each case, the basic unknowns are each a function of the time variable t.
Example 4.1
Compute the solutions of the pair of first order differential equations

$$\begin{aligned} x' &= -4x \\ y' &= 6y \end{aligned}$$

given the initial conditions $x(0) = 3$ and $y(0) = 2$.
Solution
Although we have two differential equations to solve, they are really quite separate. Thus, we need no knowledge of matrix theory to solve them.
However, the two differential equations can be written in matrix form as follows.

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

That is $\boldsymbol{X}' = A\boldsymbol{X}$, where $X' = \begin{bmatrix} x' \\ y' \end{bmatrix}$, $A = \begin{bmatrix} -4 & 0 \\ 0 & 6 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \end{bmatrix}$

Now, recall that the general solution of the differential equation
$\frac{dy}{dt} = Cy$ is $y = y_0 e^{Ct}$, where C is any constant.
Hence, the solutions of the system are $x = 3e^{-4t}$ and $y = 2e^{6t}$.
Example 4.2
Find the solution of the differential equations

$$x' = 2x - 2y + 4z$$
$$y' = 3y - 2z$$
$$z' = -y + 2z$$

with the initial conditions $x(0) = 2$, $y(0) = 0$ and $z(0) = 2$.
Solution
As we see, the system of equations here is more difficult to deal with than that of in the first Example. So, we can use our knowledge of diagonalization.

Now, set $\boldsymbol{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $A = \begin{bmatrix} 2 & -2 & 4 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$ and $\boldsymbol{X}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$

It easy to check that the characteristic polynomial of the coefficient matrix A is

$$p(\lambda) = det(\lambda I - A) = (\lambda - 2)(\lambda - 1)(\lambda - 4)$$

Hence, $\lambda_1 = 2$, $\lambda_2 = 1$ and $\lambda_3 = 4$ are the eigenvalues of A corresponding to the eigenvectors

$$\boldsymbol{v_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{v_2} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \text{ and } \boldsymbol{v_3} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$$

respectively.
Since the eigenvalue of the coefficient matrix A are distinct, the corresponding eigenvectors are linearly independent. This shows that A is diagonalizable and hence, $P = [\boldsymbol{v_1}, \ \boldsymbol{v_2}, \ \boldsymbol{v_3}]$ is the diagonalizing matrix. So, we have $P^{-1}AP = D$, that is,

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

We now introduce a new column vector of unknowns

$$\boldsymbol{Y} = \begin{bmatrix} r(t) \\ s(t) \\ q(t) \end{bmatrix}$$

through the relation $\boldsymbol{X} = P\boldsymbol{Y}$. Then, since P is a matrix of constants, we also have
$\boldsymbol{X}' = P\,\boldsymbol{Y}'$. So, $\boldsymbol{X}' = A\boldsymbol{X}$ becomes $P\boldsymbol{Y}' = A\boldsymbol{X} = A(P\boldsymbol{Y})$ so that $\boldsymbol{Y}' = (P^{-1}AP)\boldsymbol{Y}$. That is,

$$\begin{bmatrix} r'(t) \\ s'(t) \\ q'(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} r(t) \\ s(t) \\ q(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 r(t) \\ \lambda_2 s(t) \\ \lambda_3 q(t) \end{bmatrix} = \begin{bmatrix} 2r(t) \\ s(t) \\ 4q(t) \end{bmatrix}$$

Now, the new system can be written as

$$r'(t) = \lambda_1 r(t) = 2r(t)$$
$$s'(t) = \lambda_2 r(t) = s(t)$$
$$q'(t) = \lambda_3 r(t) = 4q(t)$$

These equations are separate. So, the solution of this system is then given by
$r(t) = C_1 e^{\lambda_1 t}$, $s(t) = C_2 e^{\lambda_2 t}$ and $q(t) = C_3 e^{\lambda_3 t}$.
This means,
$r(t) = C_1 e^{2t}$, $s(t) = C_2 e^{t}$ and $q(t) = C_3 e^{4t}$, where C_1, C_2 and C_3 are any constants.
Once r, s and q are known, the original unknowns x, y and z can be found from the relation $X = PY$.

So, $\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} r(t) \\ s(t) \\ q(t) \end{bmatrix} \Rightarrow \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} C_1 e^{2t} \\ C_2 e^{t} \\ C_3 e^{4t} \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} C_1 e^{2t} - 2C_2 e^{t} + C_3 e^{4t} \\ C_2 e^{t} - 2C_3 e^{4t} \\ C_2 e^{t} + C_3 e^{4t} \end{bmatrix}$$

Therefore, the general solution of the system is given by

$$x(t) = C_1 e^{2t} - 2C_2 e^{t} + C_3 e^{4t}$$
$$y(t) = C_2 e^{t} - 2C_3 e^{4t}$$
$$z(x) = C_2 e^{t} + C_3 e^{4t}$$

Now, with the initial conditions $x(0) = z(0) = 2$, and $y(0) = 0$, we get the system

$$C_1 - 2C_2 + C_3 = 2$$
$$C_2 - 2C_3 = 0$$
$$C_2 + C_3 = 2$$

Solving this, we obtain $C_1 = 4$, $C_2 = \frac{4}{3}$ and $C_3 = \frac{2}{3}$.

Thus, the particular solution of the system will be

$$x(t) = 4e^{2t} - \frac{8}{3}C_2e^t + \frac{2}{3}e^{4t}$$

$$y(t) = \frac{4}{3}e^t - \frac{4}{3}C_3e^{4t}$$

$$z(x) = \frac{4}{3}e^t + \frac{2}{3}e^{4t}$$

5. CONCLUSION

The method introduced in this study is applicable for any systems of first order differential equations containing n unknowns. This approach can be extended to systems of second order differential equations. The only restriction is that the eigenvalues of the coefficient matrix A in the system $\boldsymbol{X}' = A\boldsymbol{X}$ should be distinct.

6. ACKNOWLEDGMENT

First of all my greatest gratitude goes to the almighty GOD for providing me everything. And I would like to say thank you for my friends who helped me to prepare this paper.

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