Inversion Formula for Generalized Laplace-Fractional Mellin Transform

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Abstract— **Integral transforms appears in many fields of applied mathematics, physics, and engineering. There are several kinds of integral transforms and they have wide applications in today's technology. Recently many researchers studied some properties of applications of Mellin transform in fractional sense. Mellin transform is closely connected to Laplace transform and Fourier transform. In this paper Laplace-Fractional Mellin transform is extended in distributional generalized sense. Inversion formula for the Laplace-Fractional Mellin transform is proved.**

Index Terms— **Laplace transform, Mellin Transform, Fractional Mellin Transform, Laplace-Fractional Mellin transform.**

I. INTRODUCTION

Laplace transform is widely used transform with many applications in physics and engineering. Most widely used application of Laplace transform is to solve differential equations of higher order. In physics and engineering it is use for analysis of linear time-invariant systems, also used in signal processing to access the frequency spectrum of the of the signal in consideration. Mellin transform is basic tool for analyzing the behavior of many in mathematics and mathematical physics. Mellin transform has many applications such as quantum calculus, radar classification of ship, electromagnetic, agriculture, medical stream, Navigations etc.

Signals can be expressed by functions in mathematics. In order to satisfy the requirement of signal processing, Laplace and Fourier transforms have been introduced to solve lots of physical problems. Mellin transform offers human a new way to figure problem out [2-5]. It was R.H Mellin (1854-1933) initially gave a systematic formulation of transform and its inverse. Researchers subsequently generalized the Fourier and Mellin transforms into fractional types and multidimensional types. These all transformations provide us different ways to analyze the spectra, which finally expose the properties of different signals [6].

The Fractional Mellin transform is generalization of scale covariant transform [7] and Mellin on the scale- wrapped time frequency plane [8]. An Image encryption scheme has been presented by using two structured phase masks in the Fractional Mellin transform plane of a system, employing a phase retrieval technique. Since Fractional Mellin transform in non linear integral transform, its use to enhance the systems security [9]. Many researchers have been studied such type of various transforms and their properties but there is much scope is extending double transformations to certain class of generalized functions. In this paper we introduced generalized Laplace-Fractional Mellin transform and proved inversion formula for Laplace-Fractional Mellin transform.

I. TWO-DIMENSIONAL OFFSET FRACTIONAL FOURIER TRANSFORM:

The Laplace-Fractional Mellin transform with parameter θ of $f(x,y)$ denoted by $LFrMT\{f(t,x)\}$ performs a linear operation given by the integral transform

$$LFrMT\{f(t,x)\} = F_\theta\{f(t,x)\}(s,u) = F_\theta(s,u) = \int_0^\infty \int_0^\infty f(t,x) K_\theta(t,s,x,u)\, dt\, dx$$

where

$$K_\theta(t,s,x,u) = x^{\frac{2\pi i u}{\sin\theta}} e^{\frac{\pi i}{\tan\theta}(u^2+\log^2 x) - st} \qquad 0 < \theta \le \frac{\pi}{2} \qquad \ldots\ldots (1.1)$$

II. TEST FUNCTION SPACE $LM_{r,b}^{\alpha,\beta}$:

An infinitely differentiable complex valued smooth function φ on R^n belongs to $E(R^n)$, if for each compact set $I \subset S_a$, where

$S_a = \{x : x \in R^n, |x| \le a, a > 0\}, I \in R^n$. And k be the open sets in $R_+ \times R_+$ such that

$$\gamma_{E,b,l,q}(\varphi) = \sup_{\substack{t \in k \\ x \in I}} \left| e^{bt} D_t^l D_x^q\, \varphi(t,x) \right| < \infty \ , \qquad l,q = 0,1,2 \ldots. \qquad \ldots\ldots (1.2)$$

The space $LM_{r,b}^{\alpha,\beta}$ are equipped with their natural Hausdoff locally convex topology $\mathcal{T}_{r,b}^{\alpha,\beta}$. This topology is respectively generates by the total families of seminorms $\{\gamma_{E,b,l,q}\}$ given by (1.2).

III. DISTRIBUTIONAL TWO-DIMENSIONAL LAPLACE-FRACTIONAL MELLIN TRANSFORM

Let $LM_{r,b}^{\alpha,\beta*}$ is the dual space of $LM_{r,b}^{\alpha,\beta}$. This space $LM_{r,b}^{\alpha,\beta*}$ consist of continuous linear functional on $LM_{r,b}^{\alpha,\beta}$. The distributional Laplace-Fractional Mellin transform of $f(t,x) \in E^*(R^n)$ is defined as

$$LFrMT\{f(t,x)\} = F_\theta\{f(t,x)\}(s,u) = \langle f(t,x), K_\theta(t,s,x,u)\rangle, \qquad \cdots\cdots\cdots (2.1)$$

where

$$K_\theta(t,s,x,u) = x^{\frac{2\pi i u}{\sin\theta}} e^{\frac{\pi i}{\tan\theta}(u^2+\log^2 x)-st}$$

For each fixed $t(0 < t < \infty)$, $s > 0$ and $0 < \theta \le \frac{\pi}{2}$, the right hand side of (2.1) has sense as the application of $f(t,x) \in LM_{r,b}^{\alpha,\beta*}$ to $K_\theta(t,s,x,u) \in LM_{r,b}^{\alpha,\beta}$.

IV. INVERSION FORMULA FOR GENERALIZED LAPLACE-FRACTIONAL FOURIER TRANSFORM

The Laplace-Fractional Mellin Transform is given by

$$LFRMT\{f(t,x)\}(s,u) = \int_0^\infty \int_0^\infty f(t,x)\, x^{\frac{2\pi i}{\sin\theta}-1}\, e^{\frac{\pi i}{\tan\theta}(u^2+\log^2 x)-st} dt\, dx$$

Then by inversion it is possible to recover $f(t,x)$ by means of inversion formula

$$f(t,x) = \int_0^\infty \int_0^\infty F_\theta(s,u)\overline{K_\theta}(t,s,x,u) ds\, du,$$

where

$$\overline{K_\theta}(t,s,x,u) = \frac{1}{2\pi}\int_0^\infty \int_0^\infty F_\theta(s,u)\left(\frac{i}{\sin\theta}\right) x^{\frac{-2\pi i}{\sin\theta}}\, e^{\frac{-\pi i}{\tan\theta}(u^2+\log^2 x)+st} ds\, du$$

Proof: Laplace-Fractional Mellin Transform is given by

$$LFRMT\{f(t,x)\}(s,u) = F_\theta(s,u) = \int_0^\infty \int_0^\infty f(t,x)\, K(t,s,x,u) dt\, dx$$

where

$$K_\theta(t,s,x,u) = x^{\frac{2\pi i u}{\sin\theta}} e^{\frac{\pi i}{\tan\theta}(u^2+\log^2 x)-st}$$

Therefore,

$$LFRMT\{f(t,x)\}(s,u) = \int_0^\infty \int_0^\infty f(t,x)\, x^{\frac{2\pi i}{\sin\theta}-1}\, e^{\frac{\pi i}{\tan\theta}(u^2+\log^2 x)-st} dt\, dx$$

$$F_\theta(s,u) = \int_0^\infty \int_0^\infty f(t,x)\, x^{\frac{2\pi i}{\sin\theta}-1}\, e^{\frac{\pi i}{\tan\theta}u^2}\, e^{\frac{\pi i}{\tan\theta}\log^2 x} e^{-st} dt\, dx$$

$$e^{\frac{-\pi i}{\tan\theta}u^2} F_\theta(s,u) = \int_0^\infty \int_0^\infty g(t,x)\, x^{\frac{2\pi i}{\sin\theta}-1}\, e^{-st} dt\, dx$$

where

$$g(t,x) = f(t,x)\, x^{\frac{2\pi i}{\sin\theta}-1}$$

Put $\quad s = p \quad$ and $\quad \frac{2\pi i}{\sin\theta} = q$

$$e^{\frac{-\pi i}{\tan\theta}u^2} F_\theta(s,u) = \int_0^\infty \int_0^\infty g(t,x)\, x^{q-1}\, e^{-pt} dt\, dx$$

$$e^{\frac{-\pi i}{\tan\theta}u^2} F_\theta\left(p, \frac{\sin\theta\, q}{2\pi i}\right) = LM\{g(t,x)\}(p,q)$$

Using inverse of Laplace-Mellin transform

$$g\,(t,x) = \frac{1}{4\pi^2}\int_0^{\infty}\int_0^{\infty} G(p,q)\ x^{-q}\, e^{-pt} dp\, dq$$

where

$$G(p,q) = e^{\frac{-\pi i}{\tan\theta}u^2}\, F_\theta\left(p, \frac{\sin\theta\ q}{2\pi i}\right)$$

$$f(t,x) = \frac{1}{4\pi^2}\int_0^{\infty}\int_0^{\infty} e^{\frac{-\pi i}{\tan\theta}u^2}\ e^{\frac{-\pi i}{\tan\theta}\log^2 x} F_\theta(s,u)\ x^{-\frac{2\pi i}{\sin\theta}}\, e^{st}\ \left(\frac{2\pi i}{\sin\theta}\right) ds\, du$$

$$f(t,x) = \frac{1}{2\pi}\int_0^{\infty}\int_0^{\infty} F_\theta(s,u)\left(\frac{i}{\sin\theta}\right) x^{\frac{-2\pi i}{\sin\theta}}\, e^{\frac{-\pi i}{\tan\theta}(u^2+\log^2 x)+st} ds\, du,$$

where

$$\overline{K_\theta}(t,s,x,u) = \frac{1}{2\pi}\int_0^{\infty}\int_0^{\infty} F_\theta(s,u)\left(\frac{i}{\sin\theta}\right) x^{\frac{-2\pi i}{\sin\theta}}\, e^{\frac{-\pi i}{\tan\theta}(u^2+\log^2 x)+st} ds\, du$$

V. CONCLISION

In the present work generalization of Distributional Laplace-Fractional Mellin transform is presented. Inversion Formula for Laplace-Fractional Mellin transform is proved.

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