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Perturbation of solutions of ordinary differential equations of the Second order

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***Abstract* - In this paper we give a method of perturbation of solutions of ordinary differential equations of the second order. Specifically, it is done here in order to obtain oscillatory solutions. Secondly, we give conditions under which a system of two second-order non-linear odes is reducible to the free particle equations**

$$x'' = 0, \ \ y'' = 0.$$

***Keywords* - ordinary differential equation, canonical forms.**

I.INTRODUCTION

We apply perturbation by means force $z(x)$ with various aims (oscillatoriness , monotonically limitedness by given boundaries, transformation of oscillatoriness into monotonically and vice versa, slow rise, periodicity, transformation from oscillatoriness into periodicity, locations of zero, approximate elimination of non-linearities, useful reduction or increasing of order, clearance of non-homogeneity, various sturm theorems, oscillations their zeros, amplitudes, frequencies etc).

Let there be a linear homogenous differential equation of the second order

$$y'' + c(x)y' + d(x)y = 0 \quad \text{……….(1)}$$

Where a(x) and b(x)are given continuous functions. Then, the equation has continuous solutions of a trignometrix (elliptic or hyperbolic)type[3].Lety_1, y_2be a fundamental set of solutions, considered known. Simple problem arises: $v_1 = y_1 + z(x)$

$$v_2 = y_2 + z(x) \quad \text{………..(2)}$$

And when does it exist, if a force $z(x)$is appllied to all solutions $y_{1,2}(x)$.

If $z(x) = y_{p(x)}$ is a particular solution of the corresponding nonhomogenous differential equation

$$y'' + c(x)y' + d(x)y = g(x),$$

Where $g(x)$is continuous, i,e if

$$z(x) = y_{p(x)} \ = y_2 \int \frac{g(x)}{W} y_1 dx - y_1 \int \frac{g(x)}{W} y_2 dx \,,$$

Then the procedure given in the remainder of the article can be regarded as a kind of continuation of Lagrange's method variation of constants.

Theorem 1.

The necessary condition for the perturbed solutions (2)of the Eq.(1) to provide for a new homogenous linear differential equation of the second order with regard to $v(x)$is that function $z(x)$ is not a solution of the initial equation (1).

Proof:

Based upon the theory of linear differential equations, if $u_1$and $u_2$are solutions given with (2), the equation is

$$\begin{vmatrix} u'' & u' & u \\ u_1'' & u_1' & u_1 \\ u_2'' & u_2' & u_1 \end{vmatrix} = 0$$

Or

$$W(u_1, u_2)u'' - (u_1''u_2 - u_1u_2'')u' + (u_1''u_2' - u_1'u_2'')u = 0. \quad \text{……(3)}$$

The wronskian for the given problem is easily determined

$$W(u_1, u_2) = u_1'u_2 - u_1u_2'$$
$$= (y_1'y_2 - y_2'y_1) + z'(y_2 - y_1) - zy_2' - y_1' \quad \text{………(4)}$$

And it is a linear function of z, z'.

For the linear homogenous differential equation of the second order (3)to exist, it needs to be $W \neq o.$ Let's assume the opposite, that $W(u_1, u_2) \neq 0$.There is a linear differential equation of the first order for $z(x)$, then

$$z' - \frac{y_2' - y_1'}{y_2 - y_1} z + \frac{y_1'y_2 - y_1y_2'}{y_2 - y_1} = 0 \quad \text{……(5)}$$

Whose solutions is

$$z = C(y_2 - y_1) - (y_2 - y_1) \int \frac{e^{-\int c(x)dx}}{(y_2 - y_1)^2} dx$$

And the new wronskian is, according to the known Liouville's theorem

$$W(y_1, y_2) = y_1'y_2 - y_2'y_1 = e^{-\int c(x)dx}$$

Since y_1, y_2 are some particular integrals of the Eq. (1), the difference $y_2 - y_1 = y^*$ is a particular integrals as well; after Liouville's theorem for equation of second order, the functions

$$y^* \text{ and } y^* \int \frac{e^{-\int c(x)dx}}{(y^*)^2} dx$$

are integrals of the Eq. (1) too. Therefore, for such a perturbation $z(x)$ it is not possible to from a homogenous linear differential equation of the second order by means of (2). Various actual necessary conditions could be formulated in many ways, for various desired situations, in accordance with the proposed problem.

Let the condition of the theorem 1 be accomplished , and let's from coefficients in (3) by means of minors. There is

$$u_1''u_2 - u_2''u_1 = -a(x)W(y_1, y_2) + z''(y_2 - y_1) - z(y_2'' - y_1'')$$
$$u_1''u_2' - u_2''u_1' = b(x)W(y_1, y_2) + z''(y_2' - y_1') - z'(y_2'' - y_1'').$$

and the desired linear homogenous differential equation for $V(x)$ is obtained in the normal form

$$u'' + P(x)u' + Q(x)u = 0. \quad \dots\dots\dots \ (6)$$

With coefficients

$$P(x) = \frac{c(x)e^{-\int c(x)dx} - z''(y_2 - y_1) + z(y_2'' - y_1'')}{e^{-\int c(x)dx} + z'(y_2 - y_1) - z(y_2' - y_1')} \quad \dots\dots\dots \ (7)$$

$$Q(x) = \frac{d(x)e^{-\int c(x)dx} - z''(y_2' - y_1') - z'(y_2'' - y_1'')}{e^{-\int c(x)dx} + z'(y_2 - y_1) - z(y_2' - y_1')} \quad \dots\dots\dots\dots \ (8)$$

As the denominator $W(y_1, y_2)$ is different from zero according to the Theorem 1, then $P(x)$ and $Q(x)$ are continuous if z, z', and z'' are continuous.

II.TRANSLATION OF FUNDAMENTAL SOLUTIONS ALONG 0y axis

Let there be $z = \lambda$ =const. Then $z' = z'' = 0$,and $u_1 = y_1 + \lambda, u_2 = y_2 + \lambda$. From (7) and (8) it is obtained

$$P(x) = \frac{c(x)e^{-\int c(x)dx} + \lambda(y_2'' - y_1'')}{e^{-\int c(x)dx} - \lambda(y_2' - y_1')}$$

$$Q(x) = \frac{d(x)e^{-\int c(x)dx}}{e^{-\int c(x)dx} - \lambda(y_2' - y_1')}$$

meaning that the equation

$$v'' + \frac{c(x)e^{-\int c(x)dx} + \lambda(y_2'' - y_1'')}{e^{-\int c(x)dx} - \lambda(y_2' - y_1')} v' + \frac{d(x)e^{-\int c(x)dx}}{e^{-\int c(x)dx} - \lambda(y_2' - y_1')} v = 0,$$

could be used in order to

-obtain periodical solution without zeros;
-accomplish any other avoidance of zeros;
-increase the maximum (and consequently the minimum)of the solution;
-reach of the solutions to desired supremums.

III.CANONICAL FORMS

The canonical form is important for determination of the character of the solutions i, e.Oscillatory or monotonically, existence and locations of zeros. It is known that introducing the substation

$$y = e^{-\frac{1}{2}\int c(x)dx} Y ;$$

Where Y is a new unknown function, transformation the Eq. (1) into its canonical form

$$y'' + F(x)y = 0 \quad \dots\dots\dots \ (9)$$

Where

$$F(x) = d - \frac{c'}{2} - \frac{c^2}{4}. \quad \dots\dots\dots\dots \ (10)$$

Analogue procedure applies to canonical form of (6): the substitution

$$u = e^{-\frac{1}{2}\int P(x)dx} V$$

transforms Eq.(6) to

$$u'' + \Phi(x)u = 0 \quad \dots\dots.(11)$$

Where

$$\Phi(x) = Q - \frac{P'}{2} - \frac{P^2}{4}. \quad \dots\dots..(12)$$

For continuous a and b those canonical forms have been well elaborated since the times of strum [9], and in papers [2,3]. It is interesting that in the otherwise numerous on the Eq. (1)[4, 1, 6, 7, 5]. We did not find those results and the iteration method for the determination of zeros of oscillations.

Thus, we mention our result from [2,3]. If $a(x)$ and $b(x)$ are continuous coefficients in (1),then $B(x)$ in (9) is a continuous coefficient as well, and the solutions (9) are continuous differentiable too;

case:1

If $F(x) > 0$ there are oscillatory solutions

$$Y_1 = cos_{F(x)} = 1 - \iint F(x)dx^2 + \iint B(x) \iint B(x)dx^4 - \iint B(x) \iint B(x)dx^6 + \dots.$$
$$Y_2 = x - \iint xB(x)dx^2 + \iint B(x) \iint xB(x)dx^4 - \iint B(x) \iint B(x) \iint xB(x)dx^6 + \cdots \dots\dots\dots\dots..$$

Those functions are very similar to ordinary sine and cosine, but with variable amplitudes and zeros imposed by $B(x)$.

Case:2

If $F(x) < 0$ there are monotonic hyperbolic (exponential) solutions of the Eq. (9). Those are determined by the following series –iterations

$$Y_1 = cosh_{F(x)}x \ = \ 1 + \iint F(x)dx^2 + \iint F(x) \iint F(x)dx^4 + \iint F(x) \iint F(x) \iint F(x)dx^6 + \cdots \ldots \ldots$$

$$Y_2 = sin_{F(x)}x \quad = x + \iint xB(x)dx^2 + \iint F(x) \iint xF(x)dx^4 + \iint F(x) \iint F(x) \iint xF(x)dx^6 + \cdots \ldots \ldots \ldots ..$$

The general solutions of the Eq. (1) are essentially very simple in the case of continuous coefficients

$F(x) > 0$:

$$y = e^{-\frac{1}{2}} \int c(x)dx[c_1 \, cos_{F(x)}x \ \ + c_2 sin_{F(x)} \qquad \ldots \ldots (13)$$

Or

$F(x) > 0$:

$$y = e^{-\frac{1}{2}} \int a(x)dx[c_1 \, cos_{F(x)}x \ \ + c_2 sin_{F(x)}x]; \qquad \ldots\ldots..(14)$$

The solutions represent generalization of known classical solutions of the equation with constant coefficients$y'' + F^2 y = 0$, $y'' - F^2 = 0$, F=const

Great variety of solutions of linear homogenous differential equation of the second order are given by special functions, which are in most cases with discontinuous coefficients.

Theorem 2.

1. If $c(x)$and $d(x)$are continuously differentiable coefficients in (1)
2. If $z(x)$ is a twice differentiable function in the role of perturbation of solutions $y_{1,2,}$ and
3. If wronskian $w(y_1 y_2)$ given with (4) is different from zero.

   The perturbed equation (6) has the equations

$$v = e^{\frac{-1}{2}} \int Px(x)d$$

depending only on sign of $\phi(x)$ given with (12)then

proof.

Since for the perturbed equation (6) the canonical equation (11) is of the same type, and if $P(x)$ and $Q(x)$are continuous, therefore $\phi(x)$ given with (12) is continuous ; the same forms would then apply to (6).

Case:1

if $\Phi(x) > 0$, tthere are oscillatory solutions

$$u_1 = cos_{\phi(x)}x \ \ = 1 - \iint \Phi(x)dx^2 + \iint \Phi(x) \iint \Phi(x)dx^4 \ \iint \Phi(x) \iint \Phi(x) \iint \Phi(x)dx^6 + \cdots \ldots \ldots$$

$$u_2 = sin_{\Phi(x)}x \ = x - \iint x\Phi(x)dx^2 + \iint \Phi(x) \iint x\Phi \iint x\Phi(x)dx^4 \iint \Phi(x) \iint \Phi(x) \iint x\Phi(x)dx^6 + \cdots \ldots ....$$

and the general solution is given with(15).

Case:2

If $\Phi(x) < 0$ there are monotonic solutions

$$u_1 = cos_{\Phi(x)} \ = 1 + \iint \Phi(x)dx^2 + \iint \Phi(x) \iint \Phi(x)dx^4 \ + \iint \Phi(x) \iint \Phi(x) \iint \Phi(x)dx^6 + \cdots \ldots \ldots .$$

$$u_2 = sin_{\Phi(x)} \ = x + \iint x\Phi(x)dx^2 + \iint \Phi(x) \iint x\Phi(x)dx^4 + \iint \Phi(x) \iint \Phi(x) \iint x\Phi(x)dx^6 + \cdots \ldots \ldots ..$$

and the general solution is in form(16).

Based on this theorem, it is righteous to legalize the concept of perturbation as a kind of external influence, disturbance, force .in the case of ordinary series continuity as well.

IV.CONCLUSIONS

We have analysed the perturbation by means of force $z(x)$ with various aims (oscillatoriness, monotonicity, limitedness by given boundaries, transformation of oscillatoriness into monotonicity and vice versa, slow rise, periodicity, transformation from oscillatoriness into periodicity, locations of zeros, approximate elimination of non-linearities, useful reduction or increasing of order, clearance of non-homogeneity, various sturm theorems, oscillations, their zeros, amplitudes, frequencies etc. The results from all the examples in this work show the efficiency of this procedure. We apply amended [1-8], sturn theorems [9] with exact locations of zeros.

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