Peristaltic flow of Bingham fluid in an inclined tube

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Abstract - Peristaltic pumping of a Bingham fluid through a flexible tube with permeable wall of varying cross section is studied under the approximations of long wavelength and low Reynolds number. Using appropriate boundary conditions the analytical expressions for velocity, volumetric flow rate and stream function has been derived. The effect of permeability on the pumping characteristics is discussed and presented graphically.

Key word - Bingham fluid, non-uniform tube, embryo transport, permeable wall, peristaltic flow

1 Introduction

Peristaltic transport is nature's way of moving the content within hollow muscular structures by successive contractions, expansions of muscular fibres. This mechanism is responsible for the pumping of physiological fluids through the different parts of the human body. Several solutions are available in literature for peristaltic flows of physiological fluids like blood (Lee and Fung, 1971, Srivastava and Srivastava, 1984), Spermatic fluids (Gupta and Sheshadri. 1976; Srivastava and Srivastava, 1982, 1983) etc.

Guha *et al.* (1975) studied the mechanism of spermatic fluid transport in the vas deferens. In this paper it is reported that the vas deferens in rhesus monkeys is in the form of diverging tube with a ratio of exist to inlet dimensions of approximately a four. Gupta and Seshadri (1976) investigated the peristaltic flow through non-uniform tubes and channels with particular reference to the flow of spermatic fluid in vasdeferens. Srivastava and Srivastava (1982) extended the analysis to that of a two layered model. Srivastava *et al.* (1983 a, b, c) analyzed the peristaltic transport of a physiological fluid in a non‐uniform cross‐section. This model was applied to the flow of small intestine and ductus efferentes of reproductive ducts. Peristaltic pumping of blood in small blood vessels is studied by Srivastava and Srivastava (1984) where the blood is represented as a two fluid model consisting of Casson and Newtonian fluids. Misra and Pandey (1995) made a study on the peristaltic pumping in a tapered tube. Peristaltic flow in a tapered channel is discussed by Eytan *et al.* (2001). This mathematical model is applied to embryo transport within the uterine cavity.

Tang and Fung (1975) and Gopalan (1981) discussed blood flow by considering the microscopic blood vessels as a channel with permeable walls. They called the blood space as channel and tissue space as the porous layer. Chaturani and Ranganatha (1993) developed a mathematical model for solute transfer in an ultra filtering glomerular capillary, where capillaries are taken as permeable tubes. It is well known that the blood flow in small blood vessels occurs due to peristalsis. Motivated by this fact the peristaltic pumping of a Bingham fluid through a flexible tube with permeable wall of varying cross-section under the approximations of long wave length and low Reynolds number is investigated. The effect of permeability on the pumping characteristics is discussed.

2.Mathematical Formulation and Solution

Consider the peristaltic motion of a Bingham fluid in a non-uniform tube with permeable wall under long wavelength and low Reynolds number assumptions. (Fig 1) The flow in a tube is governed by Navier‐stokes equations whereas the flow in the permeable wall is described Darcy's law. The wall deformation due to infinite wave train of peristaltic waves is given by

$$R = H(Z, t) = a(Z) + b \, Sin \frac{2\pi}{\lambda}(Z - ct) \qquad (1)$$

where $a(Z) = a_0 + k Z$

where $a(Z)$ is the radius of the tube at distance ' Z ' from inlet, a_0 is the radius of the inlet, K is a constant whose magnitude depends on the lengths of the tube, exist and inlet dimensions, b is the amplitude of the wave, λ is the wave length, c is the velocity of the wave at t is the time. The flow is axisymmetric. Cylindrical polar coordinate system (R, θ, Z) is used.

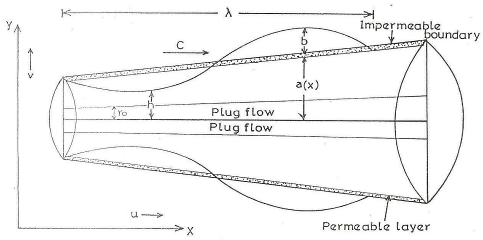

Fig. 1: Physical Model

2.1 Equation of motion

Under the assumptions that the length of the tube is an integral multiple of the wavelength λ and the pressure difference across the ends of the tube is a constant, the flow is inherently unsteady in the laboratory frame (R, θ, Z) and becomes steady in the wave frame (r, θ, Z), which is moving with velocity c along the wave. The transformation between these two frames is given by

$r = R \quad ; \theta = \Theta; z = Z - ct; \; w = W - c; \qquad \psi = \Psi \text{ ✗ } \dfrac{R^2}{2}$

and p (z) = P(Z, t) (2)

where ψ and Ψ are steam functions in the wave and laboratory frames respectively. We assume that the flow is inertia – free and the wavelength is infinite. We introduce the following non-dimensional quantities

$$\bar{r} = \frac{r}{a}; \quad \bar{z} = \frac{z}{\lambda}; \quad \bar{t} = \frac{ct}{\lambda}; \quad \bar{h} = \frac{h}{a}; \quad \phi = \frac{b}{a}; \quad \bar{F} = \frac{F}{\lambda \mu c}; \bar{\psi} = \frac{\psi}{\pi a^2 c}; \; \bar{w} = \frac{1}{r}\frac{\partial \bar{\psi}}{\partial \bar{r}}; \bar{r}_0 = \frac{r_0}{a};$$

$$\alpha = \left(\frac{a_0 m}{\sqrt{k}}\right)^{-1}; \bar{\tau}_{rz} = \frac{a\tau_{rz}}{\mu c}; \bar{\tau}_0 = \frac{a\tau_0}{\mu c}; \quad \bar{P} = \frac{a^2 P}{\lambda \mu c}; \quad \bar{q} = \frac{q}{\pi a^2 c}; \quad \bar{Q} = \frac{Q}{\pi a^2 c}$$

where $\bar{u}$ and $\bar{w}$ are the radial and axial velocities in the wave frame and $\bar{\tau}_0$ is the yield stress. Using the above non-dimensional quantities in the governing equations of motion for the inertia-free flow in the wave frame become (dropping bars)

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left\{\tau_0 - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right)\right\}\right] = \frac{-\partial p}{\partial z} \qquad (3\text{ a})$$

$$0 = \frac{\partial p}{\partial r} \qquad (3\text{ b})$$

The governing equation for the flow in the permeable wall in dimensionless form is $Q_1 = -m^2\alpha^2 \dfrac{\partial p}{\partial z}$

(3 c)

The dimensionless boundary conditions are

$\psi_p = 0$ at $r = 0$ (4)

$\psi = \psi_p$ at $r = r_0$ (5)

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) = 0 \text{ at } r = r_0 \qquad (6)$$

$$w = \frac{1}{r}\frac{\partial \psi}{\partial r} = -1 - \alpha.\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) \qquad (7)$$

The boundary condition in (7) is the Saffman (1971) slip condition at the permeable wall of the tube.

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) = \tau_0 \text{ at } r = 0 \qquad (8)$$

2.2 Solution

Solving equations (3a) and (3b) subject to the boundary conditions (4) to (8) we obtain the velocity as

$$w = \frac{Ph^2}{4}\left[1 - \frac{r^2}{4} + \frac{2\alpha}{h} - \frac{4\tau_0}{Ph}\left(1 - \frac{r}{h} + \frac{\alpha}{h}\right)\right] \qquad (9)$$

where $r_0 \le r \le h$

$$P = -\frac{\partial P}{\partial z}$$

Using the boundary condition

$$\frac{\partial w}{\partial r} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) = 0 \text{ at } r = r_0$$

the upper limit of the plug flow region is obtained as $r_0 = \dfrac{2\tau_0}{P}$ (10)

Also by using the boundary condition

$\tau_{rz} = \tau_h$ at $r = h$ (Bird *et al.* 1976)

we obtain $P = \dfrac{2\tau_h}{h}$

Hence $\frac{r_0 h}{h} = \frac{\tau_0}{\tau_h} = \tau, \quad 0 < \tau < 1$ (11)

Using the relation (2.11) and taking $r = r_0$ in equation (2.9) we get the plug flow velocity as

$$w_p = \frac{Ph^2}{4}\left[-1+\left(1-\frac{r_0}{h}\right)^2+\frac{2\alpha}{h}\left(1-\frac{r_0}{h}\right)\right] \qquad (12)$$

where $0 \le r \le r_0$

Integrating the equations (9) and (12) and using the conditions (4) and (5) we get the stream function as

$$\psi = \frac{Ph^2}{4}\left[\frac{r^2}{2}-\frac{r^4}{4h^2}-\frac{2r_0}{h}\left(\frac{r^2}{2}-\frac{r^3}{3h}+\frac{\alpha r^2}{2h}\right)+\frac{\alpha r^2}{h}+\frac{r_0^4}{12h^2}\right]-\frac{r^2}{2} \qquad (13)$$

for $r_0 \le r \le h$

$$\psi_p = \frac{Ph^2}{4}\left[\left(1-\frac{r_0}{h}\right)^2+\frac{2\alpha}{h}\left(1-\frac{r_0}{h}\right)\right]\frac{r^2}{2}-\frac{r^2}{2} \qquad (14)$$

for $0 \le r \le r_0$

The volume flux q through each cross section in the wave frame is given by

$$q = 2\int_0^{r_0} w_p r\,dr + 2\int_{r_0}^{h} w r\,dr$$

$$= \frac{ph^4}{8}\left[1-\frac{4}{3}\left(\frac{r_0}{h}\right)+\frac{1}{3}\left(\frac{r_0}{h}\right)^4+\frac{4\alpha}{h}\left(1-\frac{r_0}{h}\right)\right]-h^2 \qquad (15)$$

The instantaneous volume flow rate is given by $Q(Z,t) = 2\int_0^{r_0}(w+1)r\,dr + 2\int_{r_0}^{h}(w+1)r\,dr$

$$= q + h^2$$

$$= \frac{Ph^4}{8}\left[1-\frac{4}{3}\left(\frac{r_0}{h}\right)+\frac{1}{3}\left(\frac{r_0}{h}\right)^4+\frac{4\alpha}{h}\left(1-\frac{r_0}{h}\right)\right] \qquad (16)$$

From equation (15) we have

$$\frac{dP}{dz} = \frac{-8\left(q+h^2\right)}{h^4\left[1-\frac{4}{3}\tau+\frac{1}{3}\tau^4+\frac{4\alpha}{h}(1-\tau)\right]} \qquad (17)$$

Averaging equation (16) over one period yields the time mean flow (time-averaged volume flow rate) $\bar{Q}$ as

$$\bar{Q} = q + \frac{1}{T}\int_0^T h\,dt$$

$$= q + 1 + \frac{\varphi^2}{2} + \frac{\lambda^2 k^2 z^2}{a_0^2} + \frac{2\lambda kz}{a_0} \qquad (18)$$

2.3 The pumping characteristics

Integrating the equation (17) with respect to z over one wavelength, we get the pressure rise (drop) over one cycle of the wave as

$$\Delta p = \int_0^1 \frac{-8\left(q+h^2\right)}{h^4\left[1-\frac{4\tau}{3}+\frac{1}{3}\tau^4+\frac{4\alpha}{h}(1-\tau)\right]}dz \qquad (19)$$

The time averaged flux at zero pressure rises is denoted by $\bar{Q}_0$ and the pressure rise required to produce zero average flow rate is denoted by ΔP_0.

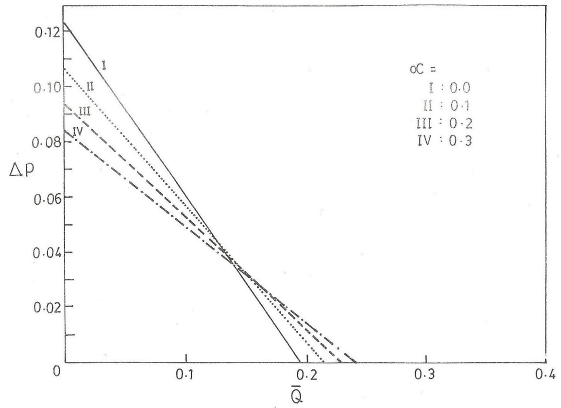

Figure:2 The variation of ΔP with $\bar{Q}$ for different values of permeability parameter α with ϕ = 0.7, λ = 20, K = 0.0018, a_0 = 0.012.

$$\Delta p_0 = \int_0^1 \frac{-8\left(1+\frac{\varphi^2}{2}-h^2\right)}{h^4\left[1-\frac{4\tau}{3}+\frac{1}{3}\tau^4+\frac{4\alpha}{h}(1-\tau)\right]}\,dz \qquad (20)$$

We find that $\bar{Q}_0$ is the independent of the yield stress. It is observed that as $\alpha \to 0$ the results deduced agree with the corresponding ones of Usha *et al.* (2002) for the peristaltic transport of a bio-fluid, in a non-uniform tube with permeable wall and as $\tau \to 0$ equations (19) and (20) reduce to the corresponding results of Jaffrin and Shapiro (1971) for the peristaltic transport of a Newtonian fluid in the tube.

The dimensionless frictional force F at the wall across one wavelength in the tube is given by

$$F = \int_0^1 h^2\left(-\frac{dp}{dz}\right)dz$$

$$= \int_0^1 \frac{8\left(q+h^2\right)}{h^2\left[1-\frac{4\tau}{3}+\frac{1}{3}\tau^4+\frac{4\alpha}{h}(1-\tau)\right]}\,dz \qquad (21)$$

3 Discussion of the results

The variation of the pressure rise ΔP with time – averaged flux $\bar{Q}$ is calculated from equation (19) for different values of permeability parameter α with $\phi = 0.7$, $\tau = 0.2$, $\lambda = 20$, $k = 0.0018$, $a = 0.012$ and is shown in figure 2. It is observed that for small values of permeability parameter the peristaltic wave passing over the tube can pump against an increased pressure rise. For fixed α, Δp decreases with increasing $\bar{Q}$. When Δp = 0, $\bar{Q}$ increases with increasing α.

The variation of time averaged flow rate with pressure rise is calculated for fixed 'α' and for different amplitude ratios ϕ with $\alpha = 0.1$; $\lambda = 20$; $k = 0.0018$, $a_0 = 0.012$; as shown in figure 3. It is noticed that the larger the amplitude ratio, the greater the pressure rise against which the pump works. For free pumping there is decrease in flux with increasing amplitude ratio ϕ.

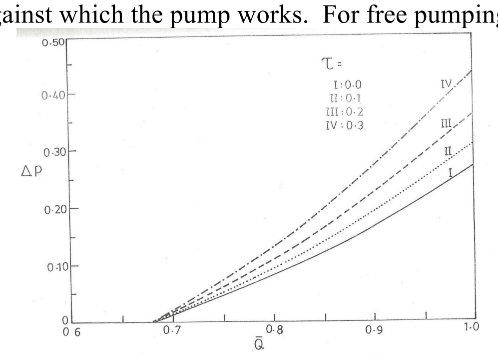

Figure:5 The variation of ΔP_0 with ϕ for different values of τ, α = 0.1, λ = 20, K = 0.0018, a_0 = 0.012.

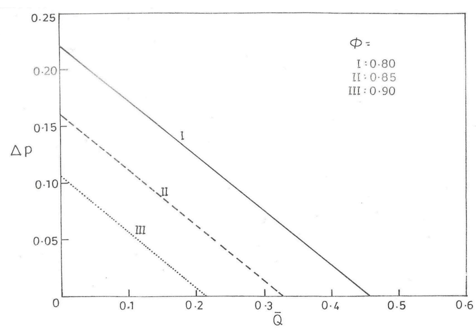

Figure:3 The variation of ΔP with $\bar{Q}$ for different values of amplitude ratios ϕ with α = 0.1, λ = 20, K = 0.0018, a_0 = 0.012

From equation (19) we have calculated Δp as a function of $\bar{Q}$ for different values of $\tau\left(=\dfrac{\tau_0}{\tau_h}\right)$ and fixed values of $\alpha = 0.1$, $\lambda = 20$, $k = 0.0018$, $\phi = 0.85$, $a_0 = 0.0012$ and is shown in figure 4. It is observed that for a given Δp, flux $\bar{Q}$ increases with increasing τ. For free pumping there is no difference in flux for Newtonian (τ=0) at Bingham fluids.

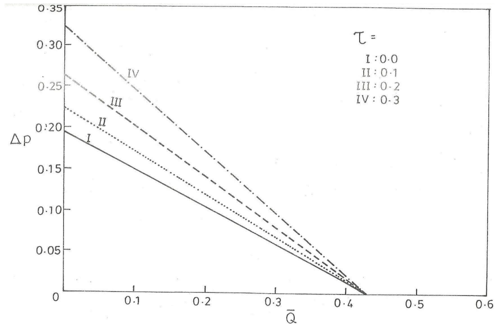

Figure:4 The variation of ΔP with $\bar{Q}$ for different values of τ with $\phi = 0.5$, $\lambda = 20$, $K = 0.0018$, $a_0 = 0.012$ and $\alpha = 0.1$.

We have calculated the pressure rise required to produce zero average flow rate Δp_0 as a function of the amplitude ratio ϕ for different values of τ and is shown in figure 5. For a Bingham fluid with a given amplitude ratio, the value of Δp_0 is larger for a Bingham fluid when compared with Newtonian fluid ($\tau = 0$). As $\phi \to 1$, Δp_0 becomes indefinitely large for a given value of τ.

From equation (21) the frictional force is computed as a function of $\bar{Q}$ for fixed $\phi = 0.7$; $\lambda = 20$; $k = 0.0018$, $a_0 = 0.012$ and for different values of permeability parameter and is depicted in figure 6. It is found that the frictional force has opposite behavior when compared with pressure rise Δp. For a given frictional force the flux $\bar{Q}$ decreases with an increase in α.

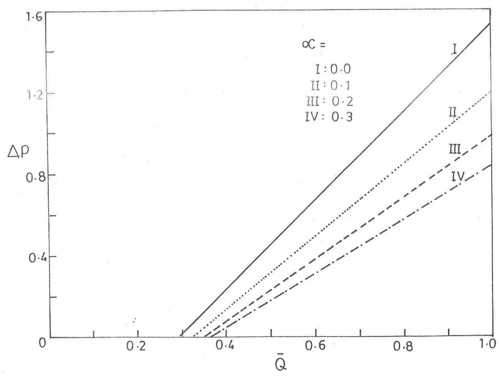

Figure:6 The variation of frictional force F with $\bar{Q}$ for different values of α with $\phi = 0.7$, $\lambda = 20$, $K = 0.0018$, $a_0 = 0.012$.

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