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# The matrix chain multiplication problem – a new approach

[1]Mrs. Analpa Ranade
[1]Student
[1]Fergusson College

*Abstract* - **Matrix chain multiplication is a classic problem in dynamic programming for ordering the chain of matrices to get the minimal number of calculations while multiplying. Here the author propose a new method to determine the multiplication sequence and bracketing which results in optimal solution. This method is easy to understand, implement and also saves time and calculations compared to the traditional method**

*keywords* - **matrix, matrix multiplication, matrix chain multiplication, dynamic programming**

## I. Introduction

Unlike scalar multiplication, matrix multiplication is not a one step process. Matrix multiplication involves a series of calculations which increase multiplicatively with the increase in number of matrices. The order in which the matrices are multiplied also affects the number of calculations. Thus before performing the actual multiplication for a chain of matrices, it is essential to first order them so as to have minimum computations during multiplication. This is called optimisation of matrix chain problem.

The traditional method (Thomas Cormen, 2002) for optimisation of matrix chain problem involves recursive method and has a time complexity $O(n^3)$. Also in the traditional method (Thomas Cormen, 2002) the actual multiplication cannot be started till the optimisation is completely done and it further reduces the efficiency as no parallel processing can be done.

To overcome this problem, I am proposing a new approach for optimisation. In this new approach, the time complexity reduces to $O(n^2)$ as against $O(n^3)$ of the traditional method. Also the new method increases the efficiency as it supports parallel processing of optimisation of the chain and actual multiplication of the matrices. **Type Style and Fonts**

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## II. Matrix multiplication

Product of two matrices is possible if and only if the number of columns of the first matrix are equal to the number of rows of the second matrix. A matrix A1 of order a0 x a1 has a0 rows and a1 columns. Let the order of matrix A2 be a1 x a2. Thus product of matrices A1 and A2 gives a matrix of order a0 x a2. This product involves a0 x a1 x a2 number of calculations, this is called the cost of the product.

## III. Matrix Chain Multiplication

Consider three matrices A1, A2, A3 having dimensions a0 x a1, a1 x a2, a2 x a3 respectively. Thus the product A1 x A2 x A3 is possible. Now to get the product of these three matrices we may follow the order:

(A1.A2).A3 or A1.(A2.A3). The total costs involved in the two cases will be different.

Eg : Let A1 have order 4 x 2, A2 has order 2 x 3 and A3 has order 3 x 5.

Then (A1.A2) will have cost= 4 x 2 x 3 = 24 and will give 4 x 3 matrix. This is then multiplied to A3 i.e. (A1. A2).A3 will have cost= 24 + 4 x 3 x 5 = 24 + 60 = 84.

Consider second case A1.(A2.A3), in this case A2.A3 will incur cost = 2 x 3 x 5 = 30 and give 2 x 5 matrix. Then A1.(A2.A3) will incur total cost = 30 + 4 x 2 x 5 = 30 + 40 = 70.

Thus to incur less cost one needs to follow the order A1.(A2.A3) this is the optimal solution.

In matrix chain multiplication problem we try to determine this order in which the multiplications should be performed or brackets should be put so as to get minimal cost and the optimal solution

## IV. Traditional Method

(Thomas Cormen, 2002): It makes use of bottom up approach by calculating the cost of matrix multiplication at each step. The distance between the matrices are varied from 1 to n-1. The cost for each pair of matrices at given distance is calculated recursively and minimum is accepted at each step. This gives the cost matrix and then from the cost matrix working backwards the bracketing has to be determined separately. This involves lengthy calculations and again backward calculation for bracketing.

## V. Proposed New Method

→ Introduction : As against the traditional method involving heavy calculations I propose a method based on finding the minimum value in an array and combining the elements giving brackets at every step to arrive at the optimal solution. To solve the matrix chain multiplication problem we use an array formed from the orders of the matrices to be multiplied and work on the array till a single matrix is left. At every step bracketing is done and also cost at every step is added to give the total cost.

- → Forming the array : Consider n matrices. The orders are taken such that chain matrix multiplication is possible for given matrices. The matrix Ai has order a(i-1) x a(i).
  From these we can form an array of orders : a(0), a(1), a(2), a(3)……… a(n) of n+1 elements.
- → Bracketing and cost calculation : At the start the initial cost is taken to be zero. From the array formed at each step we select three consecutive elements : a(i), a(j) and a(k) and eliminate the middle one i.e. a(j). When this is done then we add brackets before A(i+1) and after A(k). Also the cost goes up by the product : a(i) x a(j) x a(k). This needs to be done at every step when we eliminate one element of the array.
  Eg: id 3,5,6 are three consecutive elements selected at a step, 5 will be eliminated resulting array will be 3,6 and 3x5x6 = 90 will be added to the cost.

## VI. Algorithm

We use the following rules to eliminate the element of array

1) Find the minimum value of the array elements.
2) If in the array two minimum values are separated by a single element, eliminate the middle element and do bracketing and cost calculation as explained above. This will form a new array of length one less than the previous one. Repeat this step if possible
3) If in the array there are two minimum values separated by 2 elements, then find the max of those two elements and eliminate it, bracketing and costing need to be done. Repeat steps 2 and 3 if possible.
4) Find a remaining minimum element and combine it with 2 elements on its left or on its right whichever is possible and do bracketing and cost calculations accordingly.
   After every step check if steps 2 and 3 can be applied then apply them first, if not repeat step 4 till only three elements remain in the given array.
5) When three elements remain, only add their product to the cost. The optimal solution with bracketing and the minimum cost is ready.

## VII. Example

- → We The below example shows step by step solution for optimal matrix chain multiplication of A1.A2.A3…..A9
  Let the matrices have orders as shown in table below:

| Matrix | A1 | A2 | A3 | A4 | A5 | A6 | A7 | A8 | A9 |
|---|---|---|---|---|---|---|---|---|---|
| Order | 2 x 6 | 6 x 4 | 4 x 2 | 2 x 3 | 3 x 2 | 2 x 7 | 7 x 4 | 4 x 8 | 8 x 2 |

1) Array formed is [2,6,4,2,3,2,7,4,8,2]
2) Array [2,6,4,2,3,2,7,4,8,2]
   elements : [2,3,2] eliminate 3 and get
   A1.A2.A3.(A4.A5).A6.A7.A8.A9 the cost becomes 2x3x2 = 12
   new array is [2,6,4,2,2,7,4,8,2]
3) Array [2,6,4,2,2,7,4,8,2]
   elements : [2,6,4,2] find max of 6,4 which is 6 , eliminate 6 and get (A1.A2).A3.(A4.A5).A6.A7.A8.A9 and cost becomes = 12+2x6x4 = 60
   new array is [2,4,2,2,7,4,8,2]
4) Array [2,4,2,2,7,4,8,2]
   elements : [2,4,2] eliminate 4 and get
   ((A1.A2).A3).(A4.A5).A6.A7.A8.A9 and cost becomes = 60 + 2x4x2 = 76
   new array is [2,2,2,7,4,8,2]
5) Array [2,2,2,7,4,8,2]
   elements : [2,2,2] eliminate 2 and get
   (((A1.A2).A3).(A4.A5)).A6.A7.A8.A9 and cost becomes = 76 + 2x2x2 = 84
   new array is [2,2,7,4,8,2]
6) Array [2,2,7,4,8,2]
   elements : [2,7,4] eliminate 7 and get
   (((A1.A2).A3).(A4.A5)).(A6.A7).A8.A9 and cost becomes 84 + 2x7x4 = 140
   new array is [2,2,4,8,2]
7) Array [2,2,4,8,2]
   elements : [2,4,8,2] eliminate 8 and get
   (((A1.A2).A3).(A4.A5)).(A6.A7).(A8.A9) and cost becomes 140 + 4x8x2 = 204
   new array is [2,2,4,2]
8) Array [2,2,4,2]
   elements : [2,4,2] eliminate 4 and get
   (((A1.A2).A3).(A4.A5)).((A6.A7).(A8.A9)) and cost becomes 204 + 2x4x2 = 220
   new array is [2,2,2]
9) Array [2,2,2]
   elements : [2,2,2] : cost is 220 + 2x2x2 = 228
10) Thus for the above example the optimal solution is : (((A1.A2).A3).(A4.A5)).((A6.A7).(A8.A9)) with the minimal cost = 228

## VIII. Analysis

1) We Number of steps : For a chain of 'n' matrices, the method uses an array of n+1 elements, eliminating one element at each step, hence n-1 steps are required. In addition there is one step of finding all minimum elements of the array. Thus we can say there are total n steps to be performed which are much less as compared to the traditional method.
2) Time Complexity (O): The initial step of finding minimum takes up $T = O(n)$. The number of calculations per step are limited to traversing the array once and adding brackets and updating cost, hence max time can be taken as O(number of elements+2) per step. Since number of array elements reduce at each step,
   $T = O( n + (n+1+2)+(n+2)+(n-1+2)+(n-2+2)\ldots\ldots.(4+2))$
   In this we add the cost of the last step = 1
   $T = O(n + (n+3)+(n+2)+(n+1)+\ldots\ldots.+6+1 )= O((n^2 + 9n -16)/2) = O(n^2)$

## IX. Advantages of the new method over the traditional method

1) Easier to use as it gives both bracketing and minimal cost at same time.
2) Less calculations required, saves time and cost.
3) Time complexity $O(n^2)$ is much less compare to traditional method (Thomas Cormen, 2002) which has time complexity of order $O(n^3)$
4) In traditional method (Thomas Cormen, 2002), the entire computation needs to be completed before putting the brackets i.e. deciding the order. Hence the actual matrix multiplication can be done only after the entire chaining algorithm is completed. But in this new method, brackets are put right from the first step. Hence immediately after the first step, the actual matrix multiplication can be started. By implementing parallel algorithm of fixing the order and doing actual multiplication, the process can be speeded up considerably.

## X. Conclusion

The new proposed method is faster and easier than the traditional method (Thomas Cormen, 2002), for any number of matrices, this proposed method can be implemented in much fewer steps and doesn't need elaborate calculations. Also parallel processing of matrix multiplication makes the process much more efficient.

## References

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