Solving nonlinear partial differential equations by using sumudu decomposition method

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Abstract - In this paper, we use the sumudu decomposition method(SDM) for solving nonlinear equations. This method is a combination of the sumudu transform method and decomposition method. The nonlinear terms can be easily handled by the use of Adomian polynomials. The proposed scheme finds the solution without any discretization or restrictive assumptions and avoids the round-off errors. Several examples are presented by using this technique. The results reveal that the SDM is very efficient, simple and can be applied to other nonlinear problems.

Keywords - Sumudu decomposition method, Sumudu transform, Nonlinear partial differential equations, Adomian polynomials, Noise terms phenomena.

I. INTRODUCTION

In this work, we considers the effectiveness of the sumudu decomposition method (SDM) for solving nonlinear partial differential equations. In early 90's, Watugala [10] introduced a new integral transform, named the sumudu transform. The sumudu transform is defined over the set of functions

$$A = \{\mathrm{f}(t) \mid \exists\, M, \daleth_1, \daleth_2 > 0,\ |\mathrm{f}(t)| < M\, e^{\frac{|t|}{\daleth_j}}, \quad \text{if } t \in (-1)^j \times [0,\infty)\}$$

By the following formula

$$\bar{\mathrm{f}}(u) = S[\mathrm{f}(t)] = \int_0^\infty \mathrm{f}(ut)\, e^{-t} dt, \quad u \in (-\daleth_1, \daleth_2) \tag{1}$$

For further detail and properties of this transform, see [7-8].

Non –linear phenomena, that appear in many areas of scientific fields such as solid state physics, plasma physics, fluid mechanics, population models and chemical kinetics, can be modeled by nonlinear differential equations. The importance of obtaining the exact or approximate solutions of nonlinear partial differential equations in physics and mathematics is still a significant problem that needs new methods to discover exact or approximate solutions. Adomian Decomposition Method (ADM) [1], Homotopy Perturbation Method (HPM) [3], Variational Iteration Method (VIM) [2,5], Laplace Decomposition Method (LDM) [4,11] and Homotopy Perturbation Sumudu Transform Method (HPSTM) [9] are the various powerful mathematical methods have been proposed to obtain exact and approximate analytic solutions.

II. SUMUDU DECOMPOSITION METHOD (SDM)

To illustrate the basic idea of this method, we consider a general nonlinear non-homogenous partial differential equation with the initial conditions of the form:

$$PU(x,t) + QU(x,t) + NU(x,t) = f(x,t) \tag{2}$$

$$U(x,0) = h(x), \quad U_t(x,0) = g(x)$$

Where P is the second order linear differential operator $P = \partial^2/\partial t^2$,Q is the linear differential operator of less order than P, N represents the general nonlinear differential operator and $f(x,t)$ is the source term.

Taking the sumudu transform on both sides of equation (2), we get

$$S[PU(x,t)] + S[QU(x,t)] + S[NU(x,t)] = S[f(x,t)] \tag{3}$$

Using the differentiation property of the sumudu transform and above initial conditions, we have

$$S[U(x,t)] = u^2 S[f(x,t)] + h(x) + ug(x) - u^2 S[QU(x,t)] + NU(x,t) \tag{4}$$

Now , applying the inverse sumudu transform on both side of equation (4), we get

$$U(x,t) = F(x,t) - S^{-1}\left[u^2 S[QU(x,t) + NU(x,t)]\right] \tag{5}$$

Where $F(x,t)$ represents the term arising from the source term and the prescribed initial conditions.

The second step in sumudu decomposition method is that we represent solution as an infinite series given below

$$\mathrm{U}(x,t) = \sum_{\mathrm{n}=0}^{\infty} \mathrm{U_n}(x,t) \tag{6}$$

and the nonlinear term can be decomposed as

$$NU(x,t) = \sum_{n=0}^{\infty} A_n \tag{7}$$

Where A_n are Adomian polynomials [12] of $U_1, U_2, \dots, U_n$ and it can be calculated by formula given below

$$A_n = \frac{1}{n!}\frac{\mathrm{d^n}}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty} \lambda^i U_i\right)\right]_{\lambda=0}, n = 0,1,2, \dots \tag{8}$$

Using equation (6) and equation (7) in equation (5), we get

$$\sum_{n=0}^{\infty} U_n(x,t) = F(x,t) - S^{-1}[u^2 S[Q \sum_{n=0}^{\infty} U_n(x,t) + \sum_{n=0}^{\infty} A_n]] \quad (9)$$

On comparing both sides of the equation (9) , we get

$$U_0(x,t) = F(x,t) \quad (10)$$

$$U_1(x,t) = -S^{-1}\left[u^2 S[QU_0(x,t) + A_0]\right] \quad (11)$$

$$U_2(x,t) = -S^{-1}\left[u^2 S[QU_1(x,t) + A_1]\right] \quad (12)$$

In general the recursive relation is given by

$$U_{n+1}(x,t) = -S^{-1}\left[u^2 S[QU_n(x,t) + A_n]\right], \quad n \geq 0 \quad (13)$$

Now, we applying the sumudu transform of the right hand side of equation (13) then applying the inverse sumudu transform, we get the values of $U_0, U_1, U_2, \dots, U_n$ respectively.

III. APPLICATIONS

In order to explain the solution procedure of the Sumudu Decomposition Method (SDM), we consider the nonlinear partial differential equations.

Example 3.1. consider a nonlinear partial differential equation [6]:

$$U_t + UU_x = 2t + x + t^3 + xt^2 \quad (14)$$

With initial condition

$$U(x,0) = 0 \quad (15)$$

By applying the aforesaid method subject to the initial condition, we have

$$S[U(x,t)] = 2u^2 + xu + 6u^4 + 2xu^3 - uS[UU_x] \quad (16)$$

The inverse of sumudu transform implies that

$$U(x,t) = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3} - S^{-1}\left[uS[UU_x]\right] \quad (17)$$

Following the technique, if we assume an infinite series solution of the form (6), we obtain

$$\sum_{n=0}^{\infty} U_n(x,t) = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3} - S^{-1}[uS[\sum_{n=0}^{\infty} A_n(U)]] \quad (18)$$

Where $A_n(U)$ are Adomian polynomial [12] that represent the nonlinear terms. The first few components of Adomian polynomials, are given by

$$A_0(U) = U_0 U_{0x}$$

$$A_1(U) = U_0 U_{1x} + U_1 U_{0x} \quad (19)$$

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The recursive relation is given below

$$U_0(x,t) = t^2 + xt + \frac{t^4}{4} + \frac{xt^3}{3} \quad (20)$$

$$U_1(x,t) = -S^{-1}\left[uS[A_0(U)]\right] \quad (21)$$

$$U_{n+1}(x,t) = -S^{-1}\left[uS[A_n(U)]\right], \quad n \geq 0 \quad (22)$$

The other components of the solutions can be easily found by using above recursive relation

$$\begin{aligned} U_1(x,t) &= -S^{-1}\left[uS[A_0(U)]\right] \\ &= -S^{-1}\left[uS[U_0 U_{0x}]\right] \\ &= -\frac{1}{4}t^4 - \frac{1}{3}xt^3 - \frac{2}{15}xt^5 - \frac{7}{72}t^6 - \frac{1}{63}xt^7 - \frac{1}{98}t^8 \quad (23) \end{aligned}$$

$$\begin{aligned} U_2(x,t) &= -S^{-1}\left[uS[A_1(U)]\right] \\ &= -S^{-1}\left[uS[U_0 U_{1x} + U_1 U_{0x}]\right] \end{aligned}$$

$$= \frac{5}{8064}t^{12} + \frac{2}{2079}xt^{11} + \frac{2783}{302400}t^{10} + \frac{38}{2835}xt^9 + \frac{143}{2880}t^8 + \frac{22}{315}xt^7 + \frac{7}{12}t^6 + \frac{2}{15}xt^5 \quad (24)$$

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It is important to recall here that the noise terms appear between the components $U_0(x,t)$ and $U_1(x,t)$, where the noise terms are those pairs of terms that are identical but carrying opposite signs. More precisely, the noise terms $\pm\frac{1}{4}t^4, \pm\frac{1}{3}xt^3$ between the components $U_0(x,t)$ and $U_1(x,t)$ can be cancelled and the remaining terms of $U_0(x,t)$ still satisfy the equation. Therefore, the exact solution is given by

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t) = t^2 + xt \quad (25)$$

Example 3.2. Now, consider the following nonlinear partial differential equation [6]:

$$U_{tt} - U_t U_{xx} = -t + U \quad (26)$$

With initial conditions

$$U(x,0) = \sin x \quad (27)$$

$$U_t(x,0) = 1 \quad (28)$$

By applying the aforesaid method subject to the initial conditions, we have

$$S[U(x,t)] = u + \sin x - u^3 + u^2 S[U + U_t U_{xx}] \quad (29)$$

The inverse of sumudu transform implies that

$$U(x,t) = t + \sin x - \frac{t^3}{6} + S^{-1}\left[u^2 S[U + U_t U_{xx}]\right] \quad (30)$$

Now, applying the same procedure as in previous example we arrive at recursive relation given below

$$U_0(x,t) = t + \sin x - \frac{t^3}{6} \tag{31}$$
$$U_1(x,t) = S^{-1}[u^2 S[U_0 + B_0(U)] \tag{32}$$
$$U_{n+1}(x,t) = S^{-1}[u^2 S[U_n + B_n(U)] \,, \quad n \geq 1 \,, \tag{33}$$

Where $B_n(U)$ are Adomian polynomials [12] that represent the nonlinear terms in the above equation (33). The other components of the solutions can be easily found by using above recursive relation

$$\begin{aligned} U_1(x,t) &= S^{-1}\left[u^2 S[U_0 + B_0(U)]\right] \\ &= S^{-1}\left[u^2 S[U_0 + U_{0t}U_{0xx}]\right] \\ &= \frac{t^3}{6} - \frac{t^5}{120} + \frac{t^4}{24}\sin x \end{aligned} \tag{34}$$

It is important to recall here that the noise terms appear between the components$U_0(x,t)$ and $U_1(x,t)$, where the noise terms are those pairs of terms that are identical but carrying opposite signs. More precisely, the noise terms $\pm\frac{t^3}{6}$ between the components $U_0(x,t)$ and $U_1(x,t)$ can be cancelled and the remaining terms of $U_0(x,t)$ still satisfy the equation. Therefore, the exact solution is give by

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t) = t + \sin x \tag{35}$$

IV. CONCLUSIONS

In this work, we develops the sumudu decomposition method (SDM) for solving nonlinear problems and also solved two nonlinear partial differential equations with initial conditions. In the field of applied sciences, this technique has shown to computationally efficient in these examples that are important to many researchers. The study shows that the techniques require less computational work than existing approaches while supplying quantitatively reliable results. In conclusion, the SDM may be considered as a nice refinement in existing numerical techniques and might find the wide applications.

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