Pentagonal Graceful Labeling of Caterpillar Graphs

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Abstract - A graph G = (V, E) with p vertices and q edges is said to admit pentagonal graceful labeling if its vertices can be labeled by non negative integers such that the induced edge labels obtained by the absolute difference of the labels of end vertices are the first q pentagonal numbers. A graph G which admits pentagonal graceful labeling is called a pentagonal graceful graph. In this paper, we prove that Caterpillar is a pentagonal graceful graph.

Keywords - Pentagonal number, pentagonal graceful labeling, pentagonal graceful graph.

1. INTRODUCTION AND DEFINITIONS

The graph considered here are finite, connected, undirected and simple. The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. Terms not defined here are used in the sense of Harary [2]. For number theoretic terminology, [1] is followed. The definitions are useful for further investigations.

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. If the domain of the mapping is the set of vertices (edges / both) then the labeling is called a vertex (edge / total) labeling. There are several types of graph labeling and a detailed survey is found in [3]. The following definitions are necessary for present study.

DEFINITION 1.1 Pentagonal number is a figurate number that extends the concept of triangular and square numbers to the pentagon, but, unlike the first two, the patterns involved in the construction of pentagonal numbers are not rotationally symmetrical. The n^{th} pentagonal number, A_n is the number of distinct dots in a pattern of dots consisting of the outlines of regular pentagons with sides up to n dots, when the pentagons are overlaid so that they share one vertex. A_n is given by the formula $\frac{n(3n-1)}{2}$ for $n \geq 1$. The first few pentagonal numbers are 1, 5, 12, 22, 35, 51, 70, 92, 117, 145, 176, 210, 247, 287, 330,…

DEFINITION 1.2 Let G be a (p, q) graph. A one to one function $f: V(G) \to \{0,1,2,\dots,q\}$ is called a graceful labeling of G if the induced edge labeling f': $E(G) \to \{1,2,\dots,q\}$ defined by $f'(e) = |f(u) - f(v)|$ for each edge $e = uv$ of G is also one to one. A graph G possessing graceful labeling is called graceful graph.

DEFINITION 1.3 [4] Let G be a (p, q) graph. Let V(G), E(G) denote the vertex set and the edge set of G respectively. Consider an injective function where A_q is the q^{th} pentagonal number. That is, $A_1 = 1,\ A_2 = 5,\ A_3 = 12$ etc., $A_n = \frac{n(3n-1)}{2}$. Define the function $f^*: E(G) \to \{1, 2, \dots, A_q\}$ such that $f^*(uv) = |f(u) - f(v)|$ for all edges $uv \in E(G)$. If $f^*(E(G))$ is a sequence of distinct consecutive pentagonal numbers say $\{A_1, A_2, \dots, A_q\}$ then the function f is said to be pentagonal graceful labeling and the graph which admits such a labeling is called a pentagonal graceful graph.

2. MAIN RESULTS

Next we give some results on Caterpillars.

2.1 Definition: Let $v_1, v_2, ..., v_m$ be the m vertices of the path P_m. From each vertex v_i, i = 1,2,…,m, there are $n_i, i = 1, 2, ..., m,$ pendent vertices say $v_{i1}, v_{i2}, ..., v_{in_i}$. The resultant graph is a caterpillar and is denoted as $B(n_1, n_2, ..., n_m)$.

The graph $B(n_1, n_2)$ is called a bistar graph.

The caterpillar graph can also be defined in the following way.

G is called a caterpillar if G is a tree such that the removal of the vertices with degree 1 results in a path. And that path is called the spine of the caterpillar.

First, we prove three lemmas and then we prove a theorem on caterpillars.

Lemma1: The bistar $B(n_1, n_2)$, where $n_1 \geq 1$ and $n_2 \geq 1$ is pentagonal graceful.

Proof: Let P_2 be a path on two vertices and let v_1 and v_2 be the vertices of P_2. From v_1, there are n_1 pendent vertices say $v_{11}, v_{12}, ..., v_{1n_1}$ and from v_2, there are n_2 pendent vertices say $v_{21}, v_{22}, ..., v_{2n_2}$. The resulting graph is a bistar $B(n_1, n_2)$.

Let G = (V, E) be the bistar $B(n_1, n_2)$.

Let $V(G) = \{v_i : i = 1, 2\} \cup \{v_{1j} : 1 \leq j \leq n_1\} \cup \{v_{2j} : 1 \leq j \leq n_2\}$ and

$E(G) = \{v_1 v_2\} \cup \{v_1 v_{1j} : 1 \leq j \leq n_1\} \cup \{v_2 v_{2j} : 1 \leq j \leq n_2\}$.

Then G has $n_1 + n_2 + 1$ edges. Let $n_1 + n_2 + 1 = m$ (say).

Now label the vertices v_1, v_2 of P_2 as 0 and 1.

Then label the n_1 vertices adjacent to v_1 other than v_2 as $A_m, A_{m-1}, A_{m-2}, ..., A_{m-(n_1-1)}$. Next label the n_2 vertices adjacent to v_2 other than v_1 as $A_{m-n_1} + 1, ..., A_{m-(n_1+n_2-1)} + 1$.

We shall prove that G admits pentagonal graceful labeling.

From the definition, it is clear that $\max_{v \in V(G)} f(v)$ is A_m an $f(v) \in \{0, 1, 2, ..., A_m\}$. Also from the definition, all the vertices of G have different labeling. Hence f is one to one.

It remains to show that the edge values are of the form $\{A_1, A_2, ..., A_m\}$. The induced edge function $f^* : E(G) \rightarrow \{1, 2, ..., A_m\}$ is defined as follows.

$$f^*(v_i v_{ij}) = \begin{cases} A_{m-(j-1)} & \text{if } i = 1 \text{ and } 1 \leq j \leq n_1 \\ A_{m-(n_1+j-1)} & \text{if } i = 2 \text{ and } 1 \leq j \leq n_2 \end{cases}$$

and $f^*(v_1 v_2) = A_1$

Clearly f^* is one to one and $f^*(E(G)) = \{A_1, A_2, ..., A_m\}$. Therefore G admits pentagonal graceful labeling. Hence the graph $B(n_1, n_2)$ is pentagonal graceful

Illustration: The pentagonal graceful labeling of $B(5,4)$ is shown in figure 1.

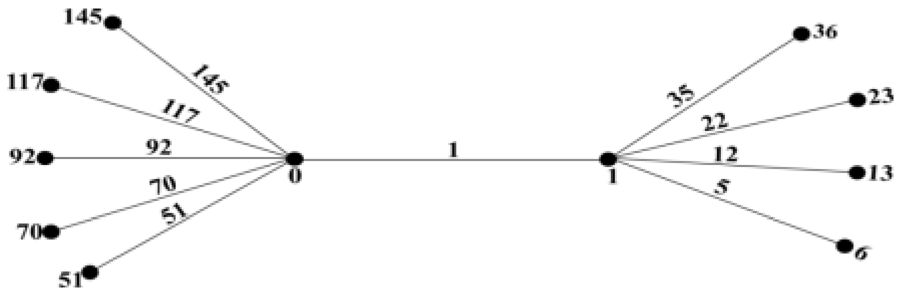

Figure 1

Lemma 2: The caterpillar $B(n_1,0,n_2)$ is pentagonal graceful for all $n_1, n_2 \geq 1$.

Proof: Let v_1, v_2, v_3 be the three vertices of P_3. From v_1, there are n_1 pendent vertices say $u_1, u_2, ..., u_{n_1}$ and from v_3, there are n_2 pendent vertices say $w_1, w_2, ..., w_{n_2}$. The resulting graph is denoted as $B(n_1,0,n_2)$. Let it be $G = (V, E)$. Then G has $n_1 + n_2 + 3$ vertices and $n_1 + n_2 + 2$ edges. Let $n_1 + n_2 + 2 = m$ (say).

Define $f : V(G) \rightarrow \{0,1,2,...,A_m\}$ as follows.

$$f(v_1) = A_m$$

$$f(v_2) = 0$$

$$f(v_3) = A_{m-n_1-1}$$

$$f(u_i) = A_m - A_{m-i}, \text{ where } 1 \leq i \leq n_1$$

$$f(w_j) = A_{m-n_1-1} + A_j, \text{ where } 1 \leq j \leq n_2.$$

We shall prove that G admits pentagonal graceful labeling. From the definition, it is clear that $\max_{v \in V(G)} f(v)$ is A_m and $f(v) \in \{0,1,2,...,A_m\}$. Also from the definition, all the vertices of G have different labeling. Hence f is one to one.

It remains to show that the edge values are of the form $\{A_1, A_2, ..., A_m\}$. The induced edge function $f^* : E(G) \rightarrow \{1,2,..., A_m\}$ is defined as follows.

$$f^*(v_1v_2) = A_m$$

$$f^*(v_2v_3) = A_{m-n_1-1}$$

$$f^*(v_1u_i) = A_{m-i}, \quad \text{where } 1 \leq i \leq n_1$$

$$f^*(v_3w_j) = A_j, \quad \text{where } 1 \leq j \leq n_2$$

Clearly f^* is one to one and $f^*(E(G)) = \{A_1, A_2, ..., A_m\}$. Therefore G admits pentagonal graceful labeling. Hence the graph $B(n_1,0,n_2)$ is pentagonal graceful for all $n_1, n_2 \geq 1$.

Illustration: The pentagonal graceful labeling of $B(5,0,4)$ is given in figure 2.

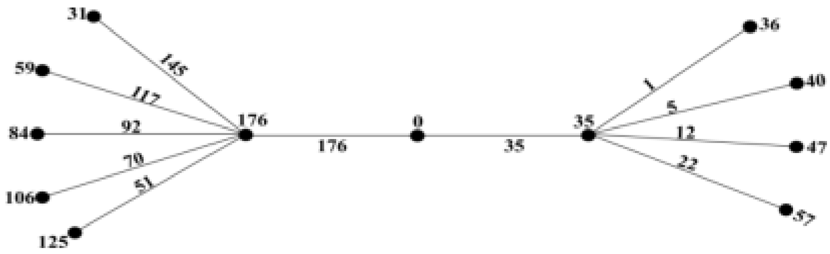

Figure 2

Lemma 3: The caterpillar $B(n_1, 1, n_2)$ is pentagonal graceful for all $n_1, n_2 \geq 1$.

Illustration: The pentagonal graceful labeling of $B(5,1,4)$ is given in figure 3.

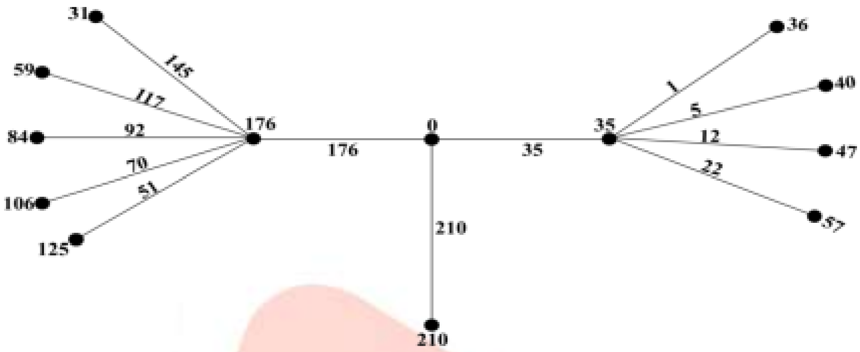

Figure 3

2.2 Theorem: Caterpillars are pentagonal graceful.

Proof: Let $v_1, v_2, \ldots, v_m$ be the m vertices of the path P_m. From each vertex v_i, i = 1,2,…,m, there are $n_i, i = 1, 2, \ldots, m,$ pendent vertices say $v_{i1}, v_{i2}, \ldots, v_{in_i}$. The resultant graph is a caterpillar and is denoted as $B(n_1, n_2, \ldots, n_m)$. Assume $m \geq 3$.

Clearly $B(n_1, n_2, \ldots, n_m)$ has $n_1 + n_2 + \ldots + n_m + (m-1)$ edges.

Let $n = n_1 + n_2 + \ldots + n_m + (m-1)$.

Define $f : V\left(B(n_1, n_2, \ldots, n_m)\right) \rightarrow \{0, 1, 2, \ldots, A_n\}$ as follows.

$$f(v_1) = 0$$

$$f(v_{1i}) = A_{n-(i-1)}, \quad \text{where } i = 1, 2, \ldots, n_1$$

$$f(v_2) = f(v_1) + A_{n-n_1}$$

$$f(v_{2i}) = f(v_2) - A_{n-n_1-i}, \quad \text{where } i = 1, 2, \ldots, n_2$$

$$f(v_3) = f(v_2) - A_{n-n_1-n_2-1}$$

$$f(v_{3i}) = f(v_3) + A_{n-n_1-n_2-1-i}, \quad \text{where } i = 1, 2, \ldots, n_3$$

$$f(v_4) = f(v_3) + A_{n-n_1-n_2-n_3-2}$$

$$f(v_{4i}) = f(v_4) - A_{n-n_1-n_2-n_3-2-i}, \quad \text{where } i = 1, 2, \ldots, n_4$$

and so on.

$$f(v_m) = \begin{cases} f(v_{m-1}) - A_{n-n_1-n_2-\cdots-n_{m-1}-(m-2)} & \text{if } m \text{ is odd} \\ f(v_{m-1}) + A_{n-n_1-n_2-\cdots-n_{m-1}-(m-2)} & \text{if } m \text{ is even} \end{cases}$$

$$f(v_{mi}) = \begin{cases} f(v_m) - A_{n-n_1-n_2-\cdots-n_{m-1}-(m-2)-i} & \text{if } m \text{ is even and } 1 \le i \le n_m \\ f(v_m) + A_{n-n_1-n_2-\cdots-n_{m-1}-(m-2)-i} & \text{if } m \text{ is odd and } 1 \le i \le n_m \end{cases}$$

$$\text{For } i = n_m,\ f(v_{mn_m}) = f(v_m) \pm A_{n-n_1-n_2-\cdots-n_{m-1}-(m-2)-n_m}$$

$$= f(v_m) \pm A_{n-n_1-n_2-\cdots-n_{m-1}-n_m-m+2}$$

$$= f(v_m) \pm A_1$$

Clearly the vertex labels are distinct and the resulting edge labels are of the form $\{A_1, A_2, ..., A_m\}$. Thus caterpillars are pentagonal graceful.

2.3 Example: The pentagonal graceful labeling of the graph B(3,2,3,5) is given in figure 4.

Here $m=4,\ n_1=3,\ n_2=2,\ n_3=3,\ n_4=5$.

Therefore, n = 3 + 2 + 3 + 5 + 3 = 16 edges.

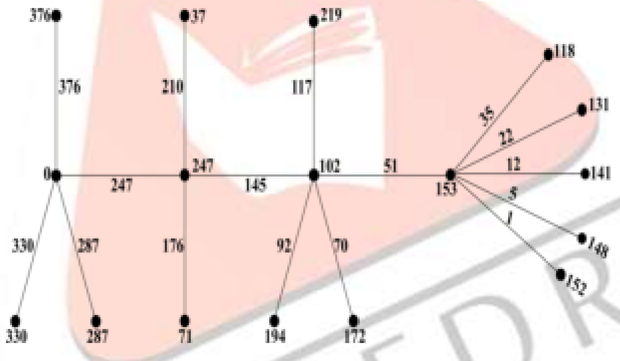

Figure 4

2.4 Conjecture: All trees are pentagonal graceful.

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