A Study Of Fuzzy Metric Space

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Abstract—The aim of this paper is to introduce some characterization of various fuzzy bounded sets in fuzzy metric spaces & some fixed-point theorems for multi valued mappings are also defined. The original concept of a KM-fuzzy metric—we call this modification by a GV-fuzzy metric. This modification allows many natural examples of fuzzy metrics, in particular, fuzzy metrics constructed from metrics. GV-fuzzy metrics appear to be more appropriate also for the study of induced topological structures. Along with the principal interest of many researchers in the theoretical aspects of the theory of fuzzy metrics—in particular, the topological and sequential properties of fuzzy metric spaces, their completeness, fixed points of mappings, etc.—fuzzy metrics have also aroused interest among specialists working in various applied areas of mathematics. Among others, fuzzy metrics have been used in decision making problems with uncertain and imprecise information and other engineering problems.

Keywords: fuzzy set, fuzzy metric space, continuous t-norm, fixed point theorems ,Cauchy sequence

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I. INTRODUCTION

A metric space is just a nonempty set *X* associated with a function *d* of two variables enabling us to measure the distance between points. In advanced mathematics, we need to find the distance not only between numbers and vectors, but also between more complicated objects like sequences, sets, and functions. In order to find an appropriate concept of a metric space, numerous approaches exist in this sphere.

In 1965, Zadeh introduced the concept of fuzzy sets, as a new way to represent the vagueness in everyday life. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, a progressive development is made in the field of fuzzy topology. One of the most important problems in fuzzy topology is to obtain an appropriate concept of fuzzy metric space. This problem has been investigated by many authors from different points of view. In particular, George and Veeramani have introduced and studied a notion of fuzzy metric space.. Kramosil and Michalek investigated the notion of fuzzy metric space which is closely related to a class of probabilistic metric spaces. George and Veeramani modified the concept of fuzzy metric space of Kramosil and Michalek, and obtained a Hausdorff and first countable topology on the modified fuzzy metric space. They also obtained a Hausdorff topology for this kind of fuzzy metric space which has very important applications in quantum particle physics, particularly in connection with both string and ∈∈∞ theory. Since then, Gregori and Romaguera proved that the topology induced by a fuzzy metric space in George and Veeramani's sense is metrizable.

In 1951, Menger introduced the concept of a statistical metric. Based on the concept of a statistical metric, Kramosil and Michalek introduced the notion of a fuzzy metric in. Here, we call it a KM-fuzzy metric. A KM-fuzzy metric is, in a certain sense, equivalent to a statistical metric, but there are essential differences in their definitions and interpretations. In 1994, George and Veeramani, slightly modified the original concept of a KM-fuzzy metric—we call this modification by a GV-fuzzy metric. This modification allows many natural examples of fuzzy metrics, in particular, fuzzy metrics constructed from metrics. GV-fuzzy metrics appear to be more appropriate also for the study of induced topological structures. Along with the principal interest of many researchers in the theoretical aspects of the theory of fuzzy metrics—in particular, the topological and sequential properties of fuzzy metric spaces, their completeness, fixed points of mappings, etc.—fuzzy metrics have also aroused interest among specialists working in various applied areas of mathematics. Among others, fuzzy metrics have been used in decision making problems with uncertain and imprecise information and other engineering problems. For example, Niskanen developed a fuzzy metric-based reasoning approach to decision making in the problems of soft computing Gregori et al. constructed a fuzzy metric that simultaneously takes into account two different distance criteria between color image pixels and used this fuzzy metric to filter noisy images, etc.The fact that KM- and GV-fuzzy metrics are obtained on the basis of a statistical metric is reflected in the assumption that the degree of closeness of two points in a fuzzy metric space corresponds to the probability of the "coincidence" of these points in the statistical metric. In particular, the fuzzy distance between two equal points is 1, while in cases when one point is "far" from the other, the fuzzy distance between them is "close" to 0. This may look strange if only one does not think of a fuzzy metric as the counterpart of a statistical metric. In these notes, we consider how the definition of a fuzzy metric can be revised in order to get a notion which is better coordinated with the intuitive meaning of a distance. Here, we restrict ourselves to the case of GV-fuzzy metrics. Obviously, a similar revision can be done for KM-fuzzy metrics as well.

II. PRELIMINARIES

To initiate the concept of a fuzzy metric space, which was introduced by Kramosil and Michalek in 1975 is recalled here.

Definition 2.1: A fuzzy set Ã is defined by Ã $=\{(x,\mu A(x)):x\in A,\mu A(x)\in[0,1]\}$.

In the pair $(x, \mu A(x))$, the first element x belongs to the classical set A, the second element $\mu A(x)$ belongs to the interval [0,1], is called the membership function.

Definition 2.2: A binary operation $*$: [0, 1] × [0, 1] → [0, 1] is a continuous t-norm if it satisfies the following conditions:

(1) $*$ is associative and commutative,

(2) $*$ is continuous,

(3) $a*1 = a$ for all $a \in [0,1]$,

(4) $a*b \le c*d$ whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$

The concept of fuzzy metric space is defined by George and Veeramani .

Examples 2.3:

Typical examples of continuous t-norm are

i. Lukasievicz t-norm: $a * b = max\{a+b-1,0\}$

ii. Product t-norm: $a * b = a.b$

iii. Minimum t-norm: $a * b = min(a,b)$

Definition 2.4: A fuzzy metric space is an ordered triple $(X,M,*)$ such that X is a nonempty set, $*$is a continuous t-norm and M is a fuzzy set on $X^2 \times[0,\infty)\to[0,1]$ satisfies the following conditions: $\forall\ x,y,z \in X$ *and* $s,t>0$

(KM1) $M(x, y, t)$ is positive , $\forall\ t>0$

(KM2) $M(x, y, t) = 1, \forall\ t > 0$ if and only if $x = y$

(KM3) $M(x, y, t) = M(y, x, t)$

(KM4) $M(x, z, t+s) \ge M(x, y, t) * M(y, z, s)$

(KM5) $M(x, y, .)$: $[0,\infty)\to[0,1]$ is left-continuous.

Then M is called a fuzzy metric on X.

In the above definition, if M (x, z, t + s) ≥ M(x, y, t) * M(y, z, s) for all t, s > 0,is replaced with M(x, z, max{ t, s}) ≥ M(x, y, t) * M(y, z, s) for all t, s > 0, then M is called non-Archimedean fuzzy metric on X.

Every non-Archimedean fuzzy metric space is itself a fuzzy metric space.

Definition 2.5: A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2\times (0, \infty) \to [0,1]$ satisfies the following conditions: $\forall\ x,y,z \in X$ *and* $s,t>0$

(GV1) $M(x, y, t)>0, \forall\ t>0$

(GV2) $M(x, y, t) = 1$ if and only if $x = y, t>0$

(GV3) $M(x, y, t) = M(y,x,t)$

(GV4) $M(x,z,t+s) \ge M(x,y,t) * M(y,z,s)$

(GV5) $M(x,y,.)$: $(0,\infty)\to[0,1]$ is continuous.

Then M is called a fuzzy metric on X.

Definition 2.6: Let $(X, M, *)$ be a fuzzy metric space, for $t>0$ the open ball $B(x,r,t)$ with a centre $x \in X$ and a radius $0 < r < 1$ is defined by $B(x,r,t)=\{y\in X\colon M(x, y, t)>1-r\}$.

A subset $A\subset X$ is called open if for each $x\in A$, there exist $t>0$ and $0<r<1$ such that $B(x,r,t)\subset A$. Let τ denote the family of all open subsets of X. Then τ is topology on X, called the topology induced by the fuzzy metric M. This topology is Hausdorff and first countable. A subset A of X is said to be F-bounded if there exist t > 0 and 0< r < 1 such that M (x , y , t) > 1- r for all x, y ∈ A.

Example 2.7: Let (X, d) be a metric space. Define $a*b=ab$ $(a*b=min\{a,b\})$ for all $a,b\in[0,1]$, and define M:$X\times X\times(0,\infty)\to[0,1]$ as $M(x, y, t)=t/(t+d(x,y))$ $\forall\ x,y,z \in X$ *and* $t>0$. Then $(X, M,*)$ is a fuzzy metric space. We call this fuzzy metric induced by the metric d the standard fuzzy metric.

Example 2.8: Let (X,d) be an ordinary metric space and φbe an increasing and continuous function from R+, into (0,1) such that $\log_{t\to\infty}\varphi(t) = 1$. Four typical examples of these functions are $\varphi(t)$=t / (t +1) , $\varphi(t)$=sin (πt / (2t +1) , $\varphi(t)$=1-e^{-t} , and $\varphi(t)=e^{-1/t}$.

Let $a*b \le ab$ for all a,b ∈ [0,1]. For each t ∈ (0,∞), define M (x, y ,t)=$[\varphi(t)]^{d(x,y)}$for all x , y ∈X . It is easy to see that (X,M,*) is a non-Archimedean fuzzy metric space.

Definition 2.9: Let $(X,M,*)$ be a fuzzy metric space

(i) A sequence $\{x_n\}$in X is said to be convergent to a point $x\in X$ if $\log_{n\to\infty} M(x,y,t)$=1 for all $t > 0$.

(ii) A sequence $\{x_n\}$in X is called a M-Cauchy sequence, if for each $0<\epsilon<1$ and $t>0$, there exists $n_0\in\mathbb{N}$ such that $M(x_n, x_m,t)>1-\epsilon$ for each $n,m\ge n0$

(iii) A fuzzy metric space $(X,M,*)$ is called M-complete if every M-Cauchy sequence is convergent .

(iv) A fuzzy metric space in which every sequence has a convergent subsequence is said to be compact.
Proposition 2.10: In a fuzzy metric space (X,M,*) if a*a ≥ a for all a ∈ [0,1] then a*b = min { a ,b } for all a , b ∈[0,1].
Lemma 2.11 Let $(X,M,*)$ be a non-Archimedean fuzzy metric space with a*b ≥ $\max(a+b-1, 0)$ & φ: X x [0,∞) →[0,1) Define the relation " ≤" on X as follows : x ≤ y ⇔ M (x, y ,t) ≥ 1 + φ(x, t) – φ (y , t) , For all t > 0. Then ≤ is a partial order on x
Lemma 2.12: Let $(X,M,*)$ be a fuzzy metric space. For all u,$v \in X$,$M(u,v,.)$ is non-decreasing function.
Proof: If $M(u,v,t)>M(u,v,s)$ for some $0<t<s$.
Then $M(u,v,t)*M(v,v,s-t)\leq M(u,v,s)<M(u,v,t)$,
Thus,$M(u,v,t)<M(u,v,t)<M(u,v,t)$, (since $M(v,v,s-t)=1$)
which is a contradiction.

3. MAIN RESULTS

Theorem 3.1: Let $(X, M, *)$ be a complete fuzzy metric space and T: $X\rightarrow X$ is continuous function and satisfied the condition: $M(Tu, Tv, t)\geq min\{M(Tu,v,t),M(u,Tv,t),M(u,v,t)\}$
Moreover, the fuzzy metric $M(u,v,t)$ satisfies the condition $\lim_{t\to\infty} M(u,v,t)=1$
Where $u,v\in X$and $u\neq v$, then T has a fixed point in X.

Theorem 3.2: In a fuzzy metric space every compact set is closed and F-bounded.
Theorem 3.3. In a fuzzy metric space every compact set is complete.
Corollary 3.4. Every closed subset of a complete fuzzy metric space is complete**.**
Lemma 3.5. Let (X, M, *) be a fuzzy metric space and let λ ∈ [0,1) then there exists a fuzzy metric m on X such that m(x,y,t) ≥ λ , for each x,y ∈ X and t >0 and m and M induce the same topology on X.

Definition 3.6. Let (X, M,*) be a fuzzy metric space, x∈X and $\emptyset \neq A\subseteq X$. We define D(x,A, t) = sup {M(x,y,t):y∈A }(t > 0) then D(x,A,t) is called a degree of closeness of x to A at t.
Definition 3.7. A topological space is called a (topologically complete) fuzzy metrizable space if there exists a (topologically complete) fuzzy metric inducing the given topology on it.
Example 3.8. Let X= (0,1]. The fuzzy metric space (X, M,.) where $M(x,y,t)=\frac{t}{t+|x-y|}$ (standard fuzzy metric) is not complete because the Cauchy sequence {1/n} in this space is not convergent. Now, if we consider triple (X,m,.) where $m(x,y,t)=\frac{t}{t+|x-y|+|\frac{1}{x}-\frac{1}{y}|}$ It is straightforward to show that (X,m,.) is a fuzzy metric space, and that is complete. Since, x_n tend to x with respect to fuzzy metric M if and only if x_n tend to x with respect to fuzzy metric m, then M and m are equivalent fuzzy metrics. Hence the fuzzy metric space (X, M,.) is topologically complete fuzzy metrizable.
Lemma 3.9. Fuzzy metrizability is preserved under countable Cartesian product.
Theorem 3.10. An open subspace of a complete fuzzy metrizable space is a complete fuzzy metrizable space.
Theorem 3.11. (Alexandroff) A set in a complete fuzzy metric space is a topologically complete fuzzy metrizable space.

II. CONCLUSION

The main contribution of this manuscript is to introduce various theorems used in fuzzy metric space .

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